Perturbation expansion in phase-ordering kinetics: I. Scalar order parameter

Gene F. Mazenko
The James Franck Institute and the Department of Physics, The University of Chicago, Chicago, Illinois 60637
(Received 3 March 1998)

A consistent perturbation theory expansion is presented for phase-ordering kinetics in the case of a nonconserved scalar order parameter. At zeroth order in this expansion, one obtains the theory due to Ohta, Jasnow, and Kawasaki (OJK). At the next nontrivial order in the expansion, worked out in d dimensions, one has small corrections to the OJK result for the nonequilibrium exponent λ and the introduction of a new exponent ν governing the algebraic component of the decay of the order parameter scaling function at large scaled distances. [S1063-651X(98)10108-3]

PACS number(s): 05.70.Ln, 64.60.Cn, 64.60.My, 64.75.+g

I. INTRODUCTION

Significant progress has been made in the theory of phase-ordering kinetics [1] using methods that introduce auxiliary fields that are taken to have Gaussian statistics. These theories well describe the qualitative scaling features of ordering in unstable systems. The methods developed by Ohta, Jasnow, and Kawasaki (OJK) [2] and the theory of unstable growth (TUG) [3] each have appealing aspects and separately give good descriptions of different aspects of the ordering problem. A major lingering question is why these methods work as well as they do, and how they can be reconciled and improved. Thus there has been a search [4–6] for a field theory description where to OJK or TUG is the zeroth-order approximation in some systematic expansion. Such an expansion is presented in this paper [7]. It has the OJK result as its zeroth-order approximation. More importantly it indicates how one goes forward to improve on these theories. The case of a scalar order parameter is treated in this paper. These ideas are generalized in a companion paper to the case of the n-vector model, and systems with continuous symmetry in the disordered state.

The problem of interest is the restoration of equilibrium in a system rendered unstable by a rapid temperature quench to a regime where the final state corresponds to a broken discrete symmetry. The ordering is controlled in this case by the decreasing area of domain walls separating the competing final degenerate states. The two coexisting theories, the OJK theory and TUG, have been useful in understanding certain aspects of this problem. TUG has led to nontrivial expressions for the nonequilibrium exponent λ, defined in detail below, which are in good agreement with values known exactly or from simulation data. However, as discussed by Mazenko and Wickham [8], in the TUG approach the auxiliary-field correlation function exhibits a nonanalytic structure at short-scaled distance which leads to unphysical results when used in calculations of defect densitites [9,10] and defect velocity distributions [11,12] for systems with continuous symmetry in the disordered states. More recently, it was shown [13] that the OJK theory can be derived from the exact continuity equation for the defect densities for point and line defects, and that it leads to smooth physical results for defect properties. On the other hand, the OJK result is only compatible with rather trivial results for the exponent λ. Thus, at this point, one does not have a theory which is both smooth enough for treating defect dynamics and yet robust enough to give nontrivial results for the nonequilibrium exponent λ.

In principle, the task for the theorist in this problem may seem obvious: Linearize the order parameter equation of motion, and formally arrive at a Gaussian field theory. Then do perturbation theory in the remaining nonlinearity. This is the well known path in conventional field theory. The problem, however, becomes clear when one looks at the simplest form for the equation of motion satisfied by the order parameter ψ in dimensionless units:

$$\frac{\partial\psi}{\partial t}=\psi-\psi^3+\nabla^2\psi. \tag{1}$$

It is clear from this form that there is no dimensionless nonlinear coupling in which to expand. If one expands in the nonlinear term ψ^3, one obtains exponential growth in time (the Cahn-Hilliard theory [14] in the case where one has a conserved order parameter) which, as a zeroth-order approximation, does not include any of the basic qualitative features of the long-time ordering. The reason is that the nonlinearity is essential in stabilizing the growth in the long-time limit. Indeed the combination $\psi-\psi^3$ must become small as the system orders. A mean field theory by Langer, Bar-on, and Miller [15] is an improvement, but is ultimately flawed by the inability to treat the separation of the two characteristic lengths in the problem. The dominant scaling length $L(t)$ grows algebraically with time, and characterizes the average separation between defects in the system. The other length is the equilibrium correlation length ξ_E. Clearly, at sufficiently long times, $L(t)\gg\xi_E$. This two-length problem was addressed by Mazenko, Valls, and Zannetti [16] (MVZ), who argued that the solution to this problem is to separate the order parameter field ψ into a peak contribution σ and a fluctuating contribution $(t)u$, $\psi=\sigma+u$. Then σ is associated with ordering on the length scale $L(t)$, and u with equilibration on the length scale ξ_E. In the structure factor the σ variable is identified with the growth of a Bragg peak with width $L^{-1}(t)$, and u is identified with the Ornstein-Zernike contribution with width governed by ξ_E^{-1}. One reason for going over to the auxiliary-field method is to insure that the

Bragg peak grows with the proper weight which corresponds to the equilibrium average of the order parameter squared.

The work of OJK predated that of MVZ, but fits into the picture developed there. OJK ignored the fluctuations ($u=0$), and assumed σ is a function of a Gaussian auxiliary field which is diffusive in nature. There have been a number of subsequent papers [17–19] clarifying the nature of the OJK result. More recently, in the work of Bray and Humayun [4], there has been an effort to derive the OJK theory in a systematic fashion. This approach will be discussed below in Sec. III.

An alternative implementation of the ideas of MVZ was carried out in the TUG. In this case, as discussed below, the mapping of the order parameter onto an auxiliary field m is motivated by the idea that m measures the distance to the nearest defect. Coupled with the assumption that the auxiliary field is Gaussian, the equation of motion satisfied by σ is enforced, and one obtains a theory describing most of the features satisfied by the order parameter scaling function.

All of these theories with auxiliary fields satisfying Gaussian statistics have been lumped together as *Gaussian closure approximations* [20]. It has been clear for some time that we need to have theories which go further [21,22]. Previous efforts at post-Gaussian approximations [5,6] had some success, but are difficult to control.

How does one construct a perturbation theory expansion for the auxiliary field m? This theory will be unusual since, on average, m is growing with time. Thus standard polynomial nonlinearities in m would be a problem. As discussed in Sec. III, there are some difficult technical problems in developing on expansion method which is self-consistent. The resolution to this problem is first to choose the proper introduction of the auxiliary field m, and then organize the treatment of the associated nonstandard nonlinear field theory through the introduction of an expansion parameter ϕ_p. At zeroth order in this parameter, one obtains the theory due to OJK, and at second order one finds expressions for the corrections to the nontrivial exponents characterizing the ordering in these systems. This parameter ϕ_p is associated with the unusual nature of the nonlinearity in the problem. Instead of a polynomial nonlinearity, one has the *sign* of the field giving the driving nonlinear terms in the equation of motion for the auxiliary field. This expansion in ϕ_p appears well behaved order by order in perturbation theory, and in some ways is similar to the bare expansion in the quartic coupling in ϕ^4 field theory. One has, for example, the possibility of resummation or use of renormalization group methods. We appear to be fortunate in this case since lower-order approximations appear to work well, give reasonable results for anomalous dimensions, without the need for extensive reorganization. One encounters a rather straightforward exponentiation of logarithmic divergences. Unlike in critical phenomena, the logs appear for all dimensionality d. In this case they are driven by internal time integrations.

A key assertion in the theories developed previously is that the auxiliary field can be treated as having Gaussian statistics. It should be understood that a theory with this feature at zeroth order must have the property that all of the higher-order cumulants for the field m must vanish at this order. It is shown here that indeed the n-point cumulant is of $O[(n/2)-1]$ in the expansion parameter. Thus the two-point cumulant G_2, which enters the determination of the order parameter correlation function at lowest order, is of $O(0)$, as expected. The four-point cumulant is of $O(1)$, and so on. This has the consequence that any function of the auxiliary field m can be expressed in terms of these cumulants, and evaluated in perturbation theory.

Turning to the question of the existence of a small parameter, it appears that the most direct connection of the expansion in ϕ_p to a more conventional expansion parameter is to the $1/n$ expansion in the n-vector model. This will be discussed in detail in paper II in this series. This expansion does not appear to be related to a $1/d$ expansion [23]. Thus the method developed Ref. [24] here has more in common with methods in critical phenomena, where one works in the spatial dimension of interest, as compared, for example, to the ϵ expansion about four dimensions.

II. OVERVIEW

A. Setting up the problem

The system studied here is the domain-wall dynamics generated by the time-dependent Ginzburg-Landau model satisfied by a nonconserved scalar order parameter $\psi(\vec{r},t)$

$$\frac{\partial\psi}{\partial t}=-\Gamma\frac{\delta F}{\delta\psi}+\eta=K[\psi], \tag{2}$$

where Γ is a kinetic coefficient, F is a Ginzburg-Landau effective free energy assumed to be of the form

$$F=\int d^d r\left(\frac{c}{2}(\nabla\psi)^2+V(\psi)\right), \tag{3}$$

where $c>0$, and the potential V is assumed to be of the symmetric degenerate double-well form. We expect only these general properties of V will be important in what follows. η is a thermal noise which is related to Γ by a fluctuation-dissipation theorem. We assume that the quench is from a high temperature ($T_I>T_c$), where the system is disordered, to zero temperature where the noise can be set to zero ($\eta=0$). It is believed [1] that our final results are independent of the exact nature of the initial state, provided it is a disordered state with short-ranged correlations.

If we rescale lengths and times we can put our equation of motion in the dimensionless form

$$\Lambda(1)\psi(1)=-V'[\psi(1)], \tag{4}$$

where the diffusion operator

$$\Lambda(1)=\frac{\partial}{\partial t_1}-\nabla_1^2 \tag{5}$$

is introduced along with the shorthand notation that 1 denotes $(\mathbf{r}_1,t_1)$. Equation (1) is just the special case of the potential $V=-\frac{1}{2}\psi^2+\frac{1}{4}\psi^4$.

B. Summary

It is well established [1] that for late times following a quench from a disordered to an ordered phase, the dynamics obey scaling, and the system can be described in terms of a single growing length $L(t)$, which is characteristic of the spacing between defects. In this scaling regime the order parameter correlation function has a universal scaling form

$$C(12)\equiv\langle\psi(1)\psi(2)\rangle=\psi_0^2\mathcal{F}(x,t_1/t_2), \tag{6}$$

where ψ_0 is the magnitude of the order parameter in the ordered phase. The scaled length x is defined as $\vec{x}=(\vec{r}_1-\vec{r}_2)/L(T)$ where, for the nonconserved order parameter case considered here, the growth law goes as $L(T)\sim T^{1/2}$, where $T=\frac{1}{2}(t_1+t_2)$. In the case of the autocorrelation function $\vec{r}_1=\vec{r}_2=\vec{r}$ we have [25,26]

$$\langle\psi(\vec{r},t_1)\psi(\vec{r},t_2)\rangle\approx\left(\frac{\sqrt{t_1t_2}}{T}\right)^{\lambda}, \tag{7}$$

where λ is a nontrivial nonequilibrium exponent, and either t_1 or t_2 is much larger than the other. At equal times and short-scaled distances,

$$\mathcal{F}(x)\equiv\mathcal{F}(x,1)=1-\alpha|x|+\cdots\ . \tag{8}$$

This nonanalytic behavior as a function of x is indicative of Porod's law [27], as conventionally given in terms of the Fourier transform

$$\mathcal{F}(Q)\approx Q^{-(1+d)} \tag{9}$$

for a large scaled wave number Q. That all of the higher-order terms in Eq. (8) are odd in $|x|$ is known as the Tomita sum rule [28]. It appears, as we will discuss below, that the large x behavior can, with proper definition of x, be put in the form

$$\mathcal{F}(x)\approx\frac{1}{x^{\nu}}e^{-(1/2)x^2} \tag{10}$$

where ν is a nontrivial subdominant exponent [29].

The main results determined in this paper is an explicit determination of $\mathcal{F}(x,t_1/t_2)$ in perturbation theory. At zeroth order, we obtain the OJK result

$$\mathcal{F}(x,t_1/t_2)=\frac{2}{\pi}\sin^{-1}[\Phi_0(t_1,t_2)e^{-(1/2)x^2}], \tag{11}$$

where

$$\Phi_0(t_1,t_2)=\left(\frac{\sqrt{t_1t_2}}{T}\right)^{\lambda_0}, \tag{12}$$

where $\lambda_0=d/2$ and $\nu_0=0$. Going to next order $O(1)$ in the expansion, discussed in detail in Sec. X, we find no change in the indices λ and ν, but quantitative changes in $\mathcal{F}(x,t_1/t_2)$. At $O(2)$, both λ and ν are shifted. The exponent λ is given at $O(2)$ by

$$\lambda=\frac{d}{2}+\omega^2\frac{2^dM_d}{3^{d/2+1}}, \tag{13}$$

where the dimensionality dependent quantities ω, K_d, and M_d are determined by

TABLE I. Values of exponent λ from the current theory, OJK, and the TUG.

Dimension	Theory	OJK	TUG	Best
1	0.6268 . . .	0.5	1	1[a]
2	1.1051 . . .	1	1.2887	1.246 ± 0.02[b]
3	1.5824 . . .	1.5	1.6726	1.838 ± 0.2[b]
Large	$d/2$	$d/2$	d/2	$d/2$[c]

[a]Exact; see Ref. [1].
[b]Numerical results from Ref. [26].
[c]Best guess.

$$2\omega+\omega^2 2^d\left(K_d+\frac{M_d}{3^{d/2+1}}\right)=1+\frac{d}{2}, \tag{14}$$

$$K_d=\int_0^1 dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}, \tag{15}$$

and

$$M_d=\int_0^1 dz\frac{z^{d/2-1}}{[1+z]^d}=\frac{1}{2}\frac{\Gamma^2(d/2)}{\Gamma(d)}. \tag{16}$$

The exponent ν governing the algebraic component of the large x behavior is given at $O(2)$ by

$$\nu=\omega^2 2^{d+1}\left(K_d+\frac{M_d}{3^{d/2+1}}\right). \tag{17}$$

At lowest order, one has the OJK results $\lambda=\frac{d}{2}$ and $\nu=0$. While K_d and ω can be worked out analytically for specific values of d, the expressions are not very illuminating. Numerical values for λ, ν and ω are given in Tables I and II along with other results for comparison [30].

III. REVIEW OF AUXILIARY FIELD METHODS

How can one organize this problem in terms of a perturbation theory expansion? Let us consider first the most direct method since the exercise is instructive and suggests other approaches. Let us ignore the fluctuation field u in the decomposition of the order parameter field, and replace the order parameter field ψ with an auxiliary field m via the mapping

$$\psi=\sigma[m], \tag{18}$$

where $\sigma[m]$, as in the TUG, satisfies the Euler-Lagrange equation for the associated stationary interface problem

TABLE II. Values of exponent ν from the current theory, OJK, and the TUG. The last column gives the values of the quantity ω.

Dimension	Theory	OJK	TUG	ω
1	1.1596 . . .	0	1	0.4601 . . .
2	1.2492 . . .	0	0.5774 . . .	0.6877 . . .
3	1.3732 . . .	0	0.3452 . . .	0.9067 . . .
Large	$d(3-2\sqrt{2})$	0	0	$\frac{d}{2}(\sqrt{2}-1)$

$$\frac{d^2\sigma}{dm^2} \equiv \sigma_2 = V'[\sigma[m]]. \tag{19}$$

In this equation, m is taken to be the coordinate. It will generally be useful to introduce the notation

$$\sigma_\ell \equiv \frac{d^\ell \sigma}{dm^\ell}. \tag{20}$$

A key point in the introduction of the field m is that the zeros of m locate the zeros of the order parameter and give the positions of interfaces in the system. Locally the magnitude of m gives the distance to the nearest defect. As a system coarsens and the distance between defects increases, the typical value of m increases linearly with $L(t)$.

Inserting the mapping [31] given by Eq. (18) into the equation of motion for ψ given by Eq. (4), one finds, after using the chain rule for differentiation, an equation for $m(1)$:

$$\sigma_1(1)\Lambda(1)m(1) = -\sigma_2(1)[1-(\nabla m(1))^2]. \tag{21}$$

To obtain an idea of the nature of this equation, take the special case of a ψ^4 potential, where one can solve for the mapping analytically and obtain the usual interfacial kink form

$$\sigma[m] = \tanh(m/\sqrt{2}). \tag{22}$$

It is easy to show that for this particular potential, $\sigma_2 = -\sqrt{2}\sigma\sigma_1$, and the equation of motion for m can be written as

$$\Lambda(1)m(1) = \sqrt{2}\sigma(m(1))[1-(\nabla m(1))^2]. \tag{23}$$

The left-hand side looks like the diffusion equation, while the right-hand side has two types of nonlinearities. The first nonlinearity is $\sigma,(m)$ which looks like sgn(m) in the bulk. The second nonlinearity is given by the $(\nabla m)^2$ term. Working along these lines, Bray and Humayun [4] made the assumption that one can make a cleaver choice of the potential and replace $-\sigma_2(m)/\sigma_1(m)$ by m, and then work with the equation of motion

$$\Lambda(1)m(1) = m(1)[1-(\nabla m(1))^2]. \tag{24}$$

This polynomial form looks promising . If one makes the assumption

$$[1-(\nabla m(1))^2] \approx \frac{\kappa}{L^2(t_1)}, \tag{25}$$

then the resulting linearized equation for m is given by

$$\Lambda(1)m(1) = \frac{\kappa}{L^2(t_1)} m(1). \tag{26}$$

This equation is formally the same as the one we will find later in the zeroth-order theory developed here, and does corresponds to the OJK theory.

It is not clear how to obtain systematic corrections to Eq. (26). Consider Eq. (24) from the point of view of dimensional analysis. Since $m \approx L$, and $\Lambda \approx L^{-2}$, one has that the left-hand side Eq. (24) is of $O(L^{-1})$, while the right-hand side is of $O(L)$. The only way that the right-hand side of Eq. (24) can be of $O(L^{-1})$ is if there is an additional constraint, given by Eq. (25), that is enforced at all orders. Formally this reduces to the statement that the nonlinear interaction

$$\mathcal{V}(1) = m(1)[1-(\nabla m)^2] - \frac{\kappa}{L^2(t_1)} m(1) \tag{27}$$

must lead to self-energy corrections which are of $O(L^{-1})$. Due to internal pairings of the ∇m terms in such an expansion, this constraint cannot be enforced order by order, and one arrives back at the dimensional analysis argument discussed above. Thus the assumption given by Eq. (25) is not self-consistent without further development [32].

One ends up concluding that the $[m(\nabla m)^2]$ nonlinearity is causing the technical problems and the OJK theory is not the zeroth-order solution of this problem *as posed*. Since there is reason to believe that the OJK theory is a good approximation of the original problem, then one might conclude that the role of the $[m(\nabla m)^2]$ nonlinearity is technical and not crucial, and one should reorganize the calculation so that it plays a less prominent role. Indeed it is useful to turn the argument around, and suggest that it is the $\sigma[m]$ nonlinearity in Eq. (23) which is important in the equation of motion for m, and that its role should be emphasized.

Let us back up a bit. Suppose, instead of the rather rigid mapping given by Eq. (18), we follow MVZ and write

$$\psi = \sigma[m] + u[m], \tag{28}$$

where $\sigma[m]$ is still the solution to the Euler-Lagrange equation (19) and $u[m]$ is to be determined. Let us substitute this mapping into the equation of motion for the order parameter, use the chain rule as in leading to Eq. (23), and obtain

$$\Lambda(1)u(1) + \sigma_1(1)\Lambda(1)m(1) = -V'[\sigma(1)+u(1)] + \sigma_2(1)[\nabla m(1)]^2. \tag{29}$$

Notice that the perspective is different here. This can be regarded as an equation for the field u. We then have the freedom to assume that m is driven by an equation of the type given by Eq. (23). However we now choose this equation such that we can find a self-consistent expansion about the OJK result. To begin, we assume this equation of motion is of the form

$$\Lambda(1)m(1) = \xi(t_1)\sigma(m(1)). \tag{30}$$

Then this is just Eq. (23) with $[1-(\nabla m)^2]$ replaced by the time-dependent quantity $\xi(t)$.

From dimensional analysis $m \approx L$, $\Lambda \approx L^{-2}$, and $\sigma \approx L^0$, so $\xi \approx L^{-1}$. These restrictions are very important. Using Eq. (30) in Eq. (29) for u leads to an equation of motion for the field $u(1)$:

$$\Lambda(1)u(1) = -V'[\sigma(1)+u(1)] + \sigma_2(1)[\nabla m(1)]^2 - \sigma_1(1)\xi(t_1)\sigma(m(1)). \tag{31}$$

The most important aspect of the solution of the last equation for $u=u[m]$ is that

$$\lim_{|m|\to\infty} u[m]=0. \tag{32}$$

Indeed, as far as the universal properties are concerned, this is almost all we need to proceed. To understand that we can construct a solution for u with this property let us consider the special case where we have a ψ^4 potential. The equation for u is given then by

$$\Lambda u+(3\sigma^2-1)u+3\sigma u^2+u^3=-\sigma_2[1-(\nabla m)^2]-\sigma_1\xi\sigma. \tag{33}$$

In the limit of large $|m|$, the derivatives of σ go exponentially to zero, and the right-hand side of Eq. (33) is exponentially small. Clearly we can construct a solution for u where it is small, and linearize the left-hand side. Remembering that $\sigma^2=1$ away from interfaces in the bulk we have

$$\Lambda u+2u=-\sigma_2[1-(\nabla m)^2]-\sigma_1\xi\ \mathrm{sgn}(m). \tag{34}$$

Notice on the left-hand side that u has acquired a mass ($=2$), and in the long-time long-distance limit the term where u is multiplied by a constant dominates the derivative terms:

$$2u=-\sigma_2[1-(\nabla m)^2]. \tag{35}$$

We have dropped the term proportional to $\xi(t)$ since it vanishes more rapidly than the other terms at large times. That the u field picks up a mass in the scaling limit can easily be seen to be a general feature of a wide class of potentials where $q_0^2=V''[\sigma=\pm\psi_0]>0$. We then have on rather general principles that the field u must vanish rapidly as one moves into the bulk away from interfaces.

One expects that the explicit construction of u is rather involved, and depends on the details of the potential chosen. If we restrict our analysis to investigating universal properties associated with bulk ordering, we will not need to know the statistics of u explicitly. If we are interested in determining interfacial properties then we need to know u in some detail. Thus, for example, if we want to determine the correlation function

$$C_{\psi^2}(12)=\langle[\psi^2(1)-\psi_0^2][\psi^2(2)-\psi_0^2]\rangle, \tag{36}$$

we will need to know the statistics of the field u. However, if we are interested in quantities like

$$C(12,\ldots,n)=\langle\psi(1)\psi(2)\cdots\psi(n)\rangle \tag{37}$$

where the points $12,\ldots,n$ are not constrained to be close together, then we do not need to know u in detail. Why is this? Consider, for example,

$$C(12)=\langle\psi(1)\psi(2)\rangle=\langle[\sigma(1)+u(1)][\sigma(2)+u(2)]\rangle.$$

The point is that because the field $u(1)$ is nonzero only near interfaces, the average $\langle u(1)\sigma(2)\rangle$ is down by a factor of $1/L^2$ relative to $\langle\sigma(1)\sigma(2)\rangle$. Out in the bulk we can use dimensional analysis to make the following estimates: $(\nabla m)^2\approx O(1)$, $\sigma_2\approx O(L^{-2})$, $\sigma_1\xi(t)\sigma\approx O(L^{-3})$, and $\Lambda u\approx O(L^{-4})$, where $\xi\approx L^{-1}$. Using these results we find that the equation of motion for u can be put into the form

$$V'[\sigma+u]=\sigma_2(\nabla m)^2. \tag{38}$$

The usefulness of this equation is that it allows us to express the equation of motion for the order parameter, in the bulk, in terms of the field m,

$$\Lambda\psi=-\sigma_2(\nabla m)^2. \tag{39}$$

We will demonstrate the usefulness of this result in Sec. XI. Our picture therefore has the m field driving the order parameter and the m field satisfying the nonlinear equation (30).

IV. FIELD THEORY FOR AUXILIARY FIELD

Let us consider the field theory associated with the equation of motion for $m(\vec{r},t)$. Our development will follow the standard Martin-Siggia-Rose (MSR) [33] method in its functional integral form as developed by DeDominicis and Peliti [34]. The analysis begins with the basic equation of motion for the field m given by Eq. (30). In the MSR method the field theoretical development requires a doubling of operators to include the field M which is conjugate to m. We also organize things so that the initial field $m_0(\vec{r})$ is also treated as an independent field. Thus it is assumed that m is zero for $t<t_0$, and one must add a term $\delta(t_1-t_0)m_0(\vec{r}_1)$ to the right-hand side of Eq. (30).

Following standard procedures, averages of interest are given as functional integrals over the fields m, M and m_0 weighted by an action A:

$$\begin{aligned}\langle m(1)m(2)\cdots m(n)M(n+1)M(n+2)\cdots M(n+\ell)\rangle\\ =\int\mathcal{D}m\mathcal{D}M\mathcal{D}m_0 m(1)m(2)\cdots m(n)\\ \times M(n+1)M(n+2)\cdots M(n+\ell)e^{A_T(m,M,m_0)}/Z,\end{aligned} \tag{40}$$

where

$$Z=\int\mathcal{D}m\mathcal{D}M\mathcal{D}m_0 e^{A(m,M,m_0)}. \tag{41}$$

The action takes the form

$$\begin{aligned}A(m,M,m_0)=-i\int d1M(1)\\ \times[\Lambda(1)m(1)-\xi(1)\sigma(1)-\delta(t_1-t_0)m_0(1)]\\ -\tfrac{1}{2}\int d^dr_1\int d^dr_2 m_0(\vec{r}_1)g^{-1}(\vec{r}_1-\vec{r}_2)\\ \times m_0(\vec{r}_2),\end{aligned} \tag{42}$$

where we use the notation $\int d1=\int dt_1 d^dr_1$, and where we assume, as is appropriate in this case, that the initial field is Gaussian and has a variance given by

$$\langle m_0(\vec{r}_1)m_0(\vec{r}_2)\rangle=g(\vec{r}_1-\vec{r}_2). \tag{43}$$

We will not have to be very specific about the form of the initial correlation function g. It will be very convenient to generate our correlation functions as functional derivatives in terms of sources which couple to the conjugate fields. Thus we introduce

$$S[h,H]=\exp\int d1[h(1)m(1)+H(1)M(1)], \tag{44}$$

and define

$$Z[h,H]=\int \mathcal{D}m\mathcal{D}M\mathcal{D}m_0 e^{A(m,M,m_0)}S[h,H]$$
$$\equiv\int \mathcal{D}m\mathcal{D}M\mathcal{D}m_0 e^{A_T(m,M,m_0)}, \tag{45}$$

where the total action is defined

$$A_T=A+\int d1[h(1)m(1)+H(1)M(1)]. \tag{46}$$

The fundamental equations of motion are given by the identities

$$\int \mathcal{D}m\mathcal{D}M\mathcal{D}m_0\frac{\delta}{\delta M(1)}e^{A_T(m,M,m_0)}=0,$$

$$\int \mathcal{D}m\mathcal{D}M\mathcal{D}m_0\frac{\delta}{\delta m(1)}e^{A_T(m,M,m_0)}=0,$$

$$\int \mathcal{D}m\mathcal{D}M\mathcal{D}m_0\frac{\delta}{\delta m_0(1)}e^{A_T(m,M,m_0)}=0,$$

which reduce to

$$\left\langle\frac{\delta}{\delta M(1)}A_T(m,M,m_0)\right\rangle_h=0,$$

$$\left\langle\frac{\delta}{\delta m(1)}A_T(m,M,m_0)\right\rangle_h=0,$$

$$\left\langle\frac{\delta}{\delta m_0(1)}A_T(m,M,m_0)\right\rangle_h=0,$$

where the subscript h indicates that the average includes the source fields h and H:

$$\langle\ldots\rangle_h=\frac{\int \mathcal{D}m\mathcal{D}M\mathcal{D}m_0 e^{A_T(m,M,m_0)}\ldots}{Z(h,H)}.$$

Taking the functional derivative with respect to m just generates the original equation of motion [Eq. (30)] with an initial condition and a source term

$$\frac{\delta}{\delta M(1)}A_T(m,M,m_0)=-i[\Lambda(1)m(1)-\xi(1)\sigma(1)$$
$$-\delta(t_1-t_0)m_0(1)]+H(1).$$

The functional derivative with respect to m is given by

$$\frac{\delta}{\delta m(1)}A_T(m,M,m_0)=i[\tilde{\Lambda}(1)M(1)+\xi(1)\sigma_M(1)]+h(1),$$

where we have introduced the quantities

$$\tilde{\Lambda}(1)=\frac{\partial}{\partial t_1}+\nabla_1^2,$$
$$\sigma_M(1)=\sigma_1(m(1))M(1). \tag{47}$$

Finally we need the derivative

$$\frac{\delta}{\delta m_0(1)}A_T(m,M,m_0)$$
$$=iM(\vec{r}_1,t_0)-\int d^dr_2 g^{-1}(\vec{r}_1-\vec{r}_2)m_0(\vec{r}_2).$$

Inserting the results of taking these derivatives into the averages, we obtain our fundamental equations:

$$-i[\tilde{\Lambda}(1)\langle M(1)\rangle_h+\xi(1)\langle\sigma_M(1)\rangle_h]=h(1), \tag{48}$$

$$i[\Lambda(1)\langle m(1)\rangle_h-\xi(1)\langle\sigma(1)\rangle_h$$
$$-\delta(t_1-t_0)\langle m_0(1)\rangle_h]=H(1), \tag{49}$$

and

$$i\langle M(\vec{r}_1,t_0)\rangle_h=\int d^dr_2 g^{-1}(\vec{r}_1-\vec{r}_2)\langle m_0(\vec{r}_2)\rangle_h. \tag{50}$$

The last equation allows one to solve for the average of the initial field in terms of the MSR field m,

$$\langle m_0(\vec{r}_1)\rangle_h=i\int d^dr_2 g(\vec{r}_1-\vec{r}_2)\langle M(\vec{r}_2,t_0)\rangle_h. \tag{51}$$

Equation (49) can then be written in the form

$$i[\Lambda(1)\langle m(1)\rangle_h-\xi(1)\langle\sigma(1)\rangle_h]$$
$$=-\int d2\Pi_0(12)\langle M(2)\rangle_h+H(1), \tag{52}$$

where

$$\Pi_0(12)\equiv\delta(t_1-t_0)\delta(t_1-t_2)g(\vec{r}_1-\vec{r}_2). \tag{53}$$

All correlation functions of interest can be generated as functional derivatives of $\langle m(1)\rangle_h$ or $\langle M(1)\rangle_h$ with respect to $h(1)$ and $H(1)$.

In the limit in which the source fields vanish, each term in the two fundamental equations vanish. Therefore, it is derivatives of these equations which are of interest. Taking the functional derivative of Eq. (48) with respect to $h(2)$ gives the equation for the response function

$$G_{Mm}(12)=\frac{\delta}{\delta h(2)}\langle M(1)\rangle_h, \tag{54}$$

with

$$-i\left[\tilde{\Lambda}(1)G_{Mm}(12)+\xi(1)\frac{\delta\langle\sigma_M(1)\rangle_h}{\delta h(2)}\right]=\delta(12). \quad (55)$$

Taking the functional derivative of Eq. (52) with respect to $H(2)$ gives

$$i\left[\Lambda(1)G_{mM}(12)-\xi(1)\frac{\delta\langle\sigma(1)\rangle_h}{\delta H(2)}\right]$$
$$=-\int d3\Pi_0(13)G_{MM}(32)+\delta(12). \quad (56)$$

Taking the functional derivative of Eq. (52) with respect to $h(2)$ gives

$$i\left[\Lambda(1)G_{mm}(12)-\xi(1)\frac{\delta\langle\sigma(1)\rangle_h}{\delta h(2)}\right]$$
$$=-\int d3\Pi_0(13)G_{Mm}(32). \quad (57)$$

Note that Eqs. (55) and (56) are redundant because of the relation

$$G_{mM}(12)=G_{Mm}(21). \quad (58)$$

Clearly we can go on and generate equations for all of the cumulants by taking functional derivatives. Let us introduce the notation that $G_{A_1,A_2,\ldots,A_n}(12\ldots n)$ is the nth-order cumulant for the set of fields $\{A_1,A_2,\ldots,A_n\}$, where field A_1 has argument (1), field A_2 has argument (2), etc. This notation is needed when we mix cumulants with m and m. As an example

$$G_{Mmmm}(1234)=\frac{\delta^3\langle m(4)\rangle_h}{\delta H(1)\delta h(2)\delta h(3)}. \quad (59)$$

As a shorthand for cumulants involving only m fields, we write

$$G_n(12\cdots n)=\frac{\delta^{n-1}}{\delta h(n)\delta h(n-1)\cdots\delta h(2)}\langle m(1)\rangle_h. \quad (60)$$

The equations governing the nth-order cumulants are given by

$$-i[\tilde{\Lambda}(1)G_{Mm\ldots m}(12\ldots n)+\hat{Q}_n(12\ldots n)]=0 \quad (61)$$

and

$$i[\Lambda(1)G_n(12\ldots n)-Q_n(12\ldots n)]$$
$$=-\int d\bar{1}\Pi_0(1\bar{1})G_{Mm\ldots m}(\bar{1}2\ldots n). \quad (62)$$

The $Q's$ are defined by

$$\hat{Q}_n(12\ldots n)=\xi(1)\frac{\delta^{n-1}}{\delta h(n)\delta h(n-1)\cdots\delta h(2)}\langle\sigma_M(1)\rangle_h, \quad (63)$$

$$Q_n(12\ldots n)=\xi(1)\frac{\delta^{n-1}}{\delta h(n)\delta h(n-1)\cdots\delta h(2)}\langle\sigma(1)\rangle_h. \quad (64)$$

With this notation the equations determining the two-point functions can be written as

$$-i[\tilde{\Lambda}(1)G_{Mm}(12)+\hat{Q}_2(12)]=\delta(12), \quad (65)$$

$$i[\Lambda(1)G_2(12)-Q_2(12)]=-\int d\bar{1}\,\Pi_0(1\bar{1})G_{Mm}(\bar{1}2). \quad (66)$$

We see that $G_{Mm\ldots m}$ and $G_{mm\ldots m}$ are coupled.

The point now is to show that there is a consistent perturbation expansion where the higher-order cumulants are also of higher order in some ordering parameter. To get started we need to express $\hat{Q}_1(12)$ and $Q_1(12)$ in terms of the fundamental cumulants of the m field. The first step in this direction is to show that these quantities can be expressed in terms of the probability distribution

$$P_h(x,1)=\langle\delta(x-m(1))\rangle_h=\frac{\langle\delta(x-m(1))S[h,H]\rangle}{Z[h,H]}, \quad (67)$$

where the averages without the subscript h are weighted by A not A_T, and the dependence on the source fields is explicit. The average over $\sigma(m)$ can then be written

$$\langle\sigma(m(1))\rangle_h=\int dx\;\sigma(x)P_h(x,1) \quad (68)$$

and

$$Q_1(1)=\int dx\;\xi(1)\sigma(x)P_h(x,1). \quad (69)$$

Evaluation of $\hat{Q}_1(1)$ is only slightly more involved. We have, using Eq. (47), that

$$\begin{aligned}\langle\sigma_M(1)\rangle_h&=\langle M(1)\sigma_1(m(1))\rangle_h\\&=\frac{1}{Z[h,H]}\frac{\delta}{\delta H(1)}\langle\sigma_1(m(1))S[h,H]\rangle\\&=\frac{1}{Z[h,H]}\frac{\delta}{\delta H(1)}\left[Z[h,H]\int dx\;\sigma_1(x)P_h(x,1)\right]\\&=\left[\langle M(1)\rangle_h+\frac{\delta}{\delta H(1)}\right]\int dx\;\sigma_1(x)P_h(x,1). \quad (70)\end{aligned}$$

Using this result in Eq. (63) with $n=1$,

$$\hat{Q}_1(1)=\int dx\;\xi(1)\sigma_1(x)\left[\langle M(1)\rangle_h+\frac{\delta}{\delta H(1)}\right]P_h(x,1).$$

Then any perturbation theory expansion for $P_h(x,1)$ will lead immediately to an expansion for $\hat{Q}_1(1)$ and $Q_1(1)$. We can then obtain $\hat{Q}_n$ and Q_n by functional differentiation.

V. PERTURBATION THEORY EXPANSION

The perturbation theory expansion for $P_h(x,1)$ is straightforward. Using the integral representation for the δ function we have

$$P_h(x,1)=\int \frac{dk}{2\pi} e^{-ikx}\Phi(k,h,1) \tag{71}$$

where

$$\Phi(k,h,1)=\langle e^{\mathcal{H}(1)}\rangle_h , \tag{72}$$

and $\mathcal{H}(1)\equiv ikm(1)$. The average of the exponential is precisely of the form which can be rewritten in terms of cumulants:

$$\Phi(k,h,1)=\exp\left[\sum_{s=1}^{\infty}\frac{1}{s!}G_{\mathcal{H}}^{(s)}(1)\right], \tag{73}$$

where $G_{\mathcal{H}}^{(s)}(1)$ is the sth-order cumulant for the field $\mathcal{H}(1)$. Since $\mathcal{H}(1)$ is proportional to $m(1)$, these are, up to factors of ik to the sth power, just the cumulants for the m field:

$$G_{\mathcal{H}}^{(s)}(1)=(ik)^s G_s(11\ldots1). \tag{74}$$

We can therefore write

$$\Phi(k,h,1)=\exp\left[\sum_{s=1}^{\infty}\frac{(ik)^s}{s!}G_s(11\ldots1)\right]. \tag{75}$$

Consider first the lowest-order contribution to Q_n, which does not vanish with the external fields h,H:

$$Q_2(12)=\int dx\ \xi(1)\sigma(x)\frac{\delta}{\delta h(2)}P_h(x,1). \tag{76}$$

We will assume, as we will show self-consistently, that in zero external field, nth order cumulants are of order $(n/2)-1$ in an expansion parameter we will develop. Expanding $\Phi(k,h,1)$ in powers of the cumulants with $n>2$, and keeping terms up to the four-point cumulant, we obtain

$$P_h(x,1)=\left[1-\frac{1}{3!}G_3(111)\frac{d^3}{dx^3}+\frac{1}{4!}G_4(1111)\frac{d^4}{dx^4}+\cdots\right]\times P_h^{(0)}(x,1) \tag{77}$$

where

$$P_h^{(0)}(x,1)=\int\frac{dk}{2\pi}\Phi_0(k,h,1)e^{-ikx} \tag{78}$$

and

$$\Phi_0(k,h,1)=e^{ikG_1(1)}e^{-(1/2)k^2G_2(11)}. \tag{79}$$

Then, after taking the derivative with respect to $h(2)$, setting the external fields to zero, and neglecting all cumulants with $n>2$, we obtain

$$\Phi_0(k,h=0,1)=e^{-(1/2)k^2S_2(1)}, \tag{80}$$

and

$$Q_2^{(0)}(12)=\int dx\ \xi(1)\sigma(x)\times\int\frac{dk}{2\pi}e^{-ikx}ikG_2(12)e^{-(1/2)k^2S_2(1)}$$

where we have defined, in zero external field,

$$S_2(1)\equiv G_2(11)=\langle m^2(1)\rangle. \tag{81}$$

There are several points to make here. Let us begin with the separation of the factor $\xi(1)\sigma(x)$ into a piece which contributes to the bulk universal properties and a part which does not. Note that in general we can write $\sigma(x)$ in the form

$$\sigma(x)=\psi_0\mathrm{sgn}(x)+\tilde{\sigma}(x), \tag{82}$$

where ψ_0 is the bulk ordered value of the magnitude of the order parameter, and $\tilde{\sigma}(x)$ goes to zero exponentially as $|x|\to\infty$. For the ψ^4 potential $\psi_0=1$, and

$$\tilde{\sigma}(x)=-\mathrm{sgn}(x)\left[\frac{2}{e^{\sqrt{2}|x|}+1}\right]. \tag{83}$$

This means that $Q_2^{(0)}(12)$ can be written as the sum of two pieces:

$$Q_2^{(0)}(12)=Q_2^{(0,B)}(12)+Q_2^{(0,N)}(12), \tag{84}$$

where

$$Q_2^{(0,B)}(12)=\xi(1)\psi_0G_2(12) \tag{85}$$

$$\times\int dx\ \mathrm{sgn}(x)\int\frac{dk}{2\pi}ike^{-ikx}e^{-(1/2)k^2S_2(1)} \tag{86}$$

and

$$Q_2^{(0,N)}(12)=G_2(12)\int dx\ \xi(1)\tilde{\sigma}(x)\times\int\frac{dk}{2\pi}ike^{-ikx}e^{-(1/2)k^2S_2(1)}.$$

In both contributions we have the integral

$$\int\frac{dk}{2\pi}e^{-ikx}ike^{-(1/2)k^2S_2(1)}=-\frac{d}{dx}\int\frac{dk}{2\pi}e^{-ikx}e^{-(1/2)k^2S_2(1)}=-\frac{d}{dx}\Phi_0(x,1), \tag{87}$$

where $\Phi_0(x,1)$ is the Fourier transform of $\Phi_0(k,h=0,1)$:

$$\Phi_0(x,1)=\int\frac{dk}{2\pi}e^{-ikx}e^{-(1/2)k^2S_2(1)}=\frac{e^{-x^2/[2S_2(1)]}}{\sqrt{2\pi S_2(1)}}. \tag{88}$$

In evaluating $Q_2^{(0,N)}(12)$ we can take the derivative of $\Phi_0(x,1)$ and expand in inverse powers of $S_2(1)$ to obtain, to leading order,

$$Q_2^{(0,N)}(12)=G_2(12)\int dx\ \xi(1)\tilde{\sigma}(x)\frac{x}{\sqrt{2\pi}S_2^{3/2}(1)},$$

where it is crucial that $\tilde{\sigma}(x)$ vanish for large $|x|$ so that the x integral exists. Turning to $Q_2^{(0,B)}(12)$, we have

$$Q_2^{(0,B)}(12)=\xi(1)\psi_0 G_2(12)\int dx\ \text{sgn}(x)\left[-\frac{d}{dx}\Phi_0(x,1)\right].$$

In this case we integrate by parts in the integral over x and use $(d/dx)\text{sgn}(x)=2\delta(x)$ to obtain

$$Q_2^{(0,B)}(12)=\xi(1)\psi_0 G_2(12)2\Phi_0(0,1)$$
$$=\xi(1)\psi_0 G_2(12)\sqrt{\frac{2}{\pi S_2(1)}}. \tag{89}$$

The key observation is that $Q_2^{(0,N)}(12)$ is down by a factor of $1/L^2$ relative to $Q_2^{(0,B)}(12)$. It should be clear that this is a general result which will hold order by order in perturbation theory. As far as the bulk ordering properties are concerned we can replace $\sigma(x)\rightarrow\psi_0\text{sgn}(x)$. Thus we could have started with the equation of motion for the field m

$$\Lambda(1)m(1)=\xi(1)\psi_0\text{sgn}(m(1)) \tag{90}$$

if we focus only on bulk ordering properties [35].

Turning to the other nonlinear quantity $\hat{Q}_2$, we have, in general,

$$\hat{Q}_2(12)=\int dx\ \xi(1)\sigma_1(x)\times\left[G_{Mm}(12)+\langle M(1)\rangle_h\frac{\delta}{\delta h(2)}+\frac{\delta^2}{\delta h(2)\delta H(1)}\right]P_h(x,1). \tag{91}$$

Clearly, term by term in our expansion we will find that the leading contribution comes from $\sigma_1(x)\rightarrow\psi_0 2\delta(x)$ with the remaining terms leading to contributions which are of higher order in $1/L$. We therefore need only consider

$$\hat{Q}_2(12)=\xi(1)\psi_0\int dx\ 2\delta(x)\times\left[G_{Mm}(12)+\langle M(1)\rangle_h\frac{\delta}{\delta h(2)}+\frac{\delta^2}{\delta h(2)\delta H(1)}\right]P_h(x,1)$$

if we are only interested in the bulk universal properties.

A key observation is that, as we analyze contributions to Q_n or $\hat{Q}_n$, we will find that each term consists of products of correlation functions and response functions with legs tied together by factors defined by

$$\phi_p(1)\equiv\int dx\ \text{sgn}(x)\int\frac{dk}{2\pi}ik^{2p+1}e^{-ikx}\Phi_0(k,1)$$
$$=2\int\frac{dk}{2\pi}k^{2p}e^{-(1/2)k^2S_2(1)}=\left(-2\frac{d}{dS_2(1)}\right)^p\phi_0(1), \tag{92}$$

where we have used an integration by parts in going from the first to the second line and defined

$$\phi_0(1)=2\int\frac{dk}{2\pi}e^{-(1/2)k^2S_2(1)}=\sqrt{\frac{2}{\pi S_2(1)}}. \tag{93}$$

Each term in the perturbation theory expansion for Q_n or $\hat{Q}_n$ will be proportional to factors of ϕ_p. The perturbation expansion is ordered by the sum of the labels p on ϕ_p. Thus a contribution with insertions $\phi_1\phi_2\phi_1$, each factor typically associated with different times, is of $O(4)$. We shall refer to this expansion as the *ϕ expansion*. It should be emphasized that at this stage that this is a *formal* expansion. At order n it is true that $\phi_p\approx L^{-(2p+1)}$ which is small; however, it will be multiplied, depending on the quantity expanded, by positive factors of $L(t)$ such that each term in the expansion in ϕ_p has the same overall leading power with respect to $L(t)$.

To see how this expansion works let us consider first the two-point quantity $Q_2(12)$, defined by

$$Q_2(12)=\int dx\,\xi(1)\text{sgn}(x)\frac{\delta P_h(x,1)}{\delta h(2)}. \tag{94}$$

Using Eqs. (71) and (75), and taking the derivatives with respect to $h(2)$, we find, in the case of zero external fields,

$$Q_2(12)=\xi(1)\int dx\ \text{sgn}(x)\int\frac{dk}{2\pi}e^{-ikx}\Phi(k,h=0,1)\times\sum_{s=0}^{\infty}\frac{(ik)^{2s+1}}{(2s+1)!}G_{2s+2}(11\ldots12). \tag{95}$$

Since all odd cumulants vanish in the case of zero external fields, and

$$\Phi(k,h=0,1)=\exp\left[\sum_{s=1}^{\infty}\frac{(-1)^sk^{2s}}{(2s)!}S_{2s}(1)\right] \tag{96}$$

where

$$S_{2s}(1)=G_{2s}(11\ldots1). \tag{97}$$

Let us define the set of vertices

$$\mathcal{V}_p(1)=\int dx\ \text{sgn}(x)\int\frac{dk}{2\pi}ik^{2p+1}e^{-ikx}\Phi(k,h=0,1), \tag{98}$$

which reduces, after following the same set of steps in reducing the original expression for ϕ_p, to

$$\mathcal{V}_p(1)=2\int\frac{dk}{2\pi}k^{2p}\Phi(k,h=0,1), \tag{99}$$

which is independent of position. Then the quantity $Q_2(12)$, which appears in the equation of motion for $G_2(12)$, is given in the form

$$Q_2(12)=\xi(1)\psi_0\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}G_{2s+2}(11\ldots12)\times\int dx\ \mathrm{sgn}(x)\int\frac{dk}{2\pi}e^{-ikx}ik^{2s+1}\Phi(k,1)$$
$$=\xi(1)\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}\mathcal{V}_s(1)G_{2s+2}(11\ldots12),\tag{100}$$

where we have used the definition of $\mathcal{V}_s(1)$ given by Eq. (98) in the last step.

It should be clear that the vertices $\mathcal{V}_s(1)$ are of at least $O(s)$ in the ϕ expansion. By direct expansion of $\Phi(k,h=0,1)$ about $\Phi_0(k,h=0,1)$ we obtain

$$\mathcal{V}_s(1)=\phi_s(1)+\frac{S_4(1)}{4!}\phi_{s+2}(1)-\frac{S_6(1)}{6!}\phi_{s+3}(1)+\phi_{s+4}(1)\left[\frac{S_4^2(1)}{2(4!)^2}+\frac{S_8(1)}{8!}\right]+\cdots.\tag{101}$$

Then, since we will find $S_{\ell}(1)\approx O[(\ell/2)-1]$, the terms in the expansion for $\mathcal{V}_s$ are of $O(s)$, $O(s+3)$, $O(s+5)$, and $O(s+6)$, respectively.

Let us turn next to $\hat{Q}_2(12)$ given by Eq. (91). In the limit of zero external fields we can set the term proportional to $\langle M(1)\rangle$ to zero, and then evaluate the second derivatives, $\delta^2P_h(x,1)/\delta h(2)\delta H(1)$. After a significant amount of algebra we obtain [36]

$$\hat{Q}_2(12)=\int dx\ 2\xi(1)\psi_0\delta(x)\int\frac{dk}{2\pi}e^{-ikx}\Phi(k,1)\left[G_{Mm}(12)+\sum_{s=1}^{\infty}\frac{(ik)^s}{s!}G^{(s+2)}_{mm\ldots mMm}(11\ldots112)\right]$$
$$=\int dx2\xi(1)\psi_0\delta(x)\int\frac{dk}{2\pi}e^{-ikx}\Phi(k,1)\sum_{s=0}^{\infty}\frac{(ik)^s}{s!}G^{(s+2)}_{mm\ldots mMm}(11\ldots112)$$
$$=\xi(1)\psi_0\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s)!}\mathcal{V}_s(1)G^{(2s+2)}_{mm\ldots mMm}(11\ldots112).\tag{102}$$

Before going on to discuss the structure of the perturbation theory at higher order, let us made sure the theory is sensible at zeroth order where, from Eqs. (100) and (102),

$$Q_2^{(0)}(12)=\xi(1)\psi_0\phi_0(1)G_2(12),\tag{103}$$

$$\hat{Q}_2^{(0)}(12)=\xi(1)\psi_0\phi_0(1)G_{Mm}(12).\tag{104}$$

VI. ZEROTH-ORDER THEORY FOR TWO-POINT CORRELATION FUNCTIONS

The equations of motion at zeroth order for the two-point correlation function is given by Eqs. (61) and (62), with $\hat{Q}_2$ and Q_2 replaced by Eqs. (104) and (103). Thus we have

$$-i[\Lambda(1)+\omega_0(1)G^{(0)}_{Mm}(12)=\delta(12),\tag{105}$$

$$i[\Lambda(1)-\omega_0(1)]G_2^{(0)}(12)=-\int d\overline{1}\Pi_0(1\overline{1})G^{(0)}_{Mm}(\overline{1}2),\tag{106}$$

where we have defined

$$\omega_0(1)=\xi(1)\psi_0\phi_0(1).\tag{107}$$

The first step in the solution to these equations is to Fourier transform over space. Taking the equation for the response function first, we obtain

$$-i\left[\frac{\partial}{\partial t_1}-q^2+\omega_0(t_1)\right]G^{(0)}_{Mm}(q,t_1t_2)=\delta(t_1-t_2).\tag{108}$$

This first-order differential equation has the solution

$$G^{(0)}_{Mm}(q,t_1t_2)=-i\theta(t_2-t_1)\exp\left[\int_{t_2}^{t_1}d\tau[q^2-\omega_0(\tau)]\right]$$
$$=-i\theta(t_2-t_1)R(t_2,t_1)e^{-q^2(t_2-t_1)},\tag{109}$$

and we have defined

$$R(t_1,t_2)=e^{\int_{t_2}^{t_1}d\tau\ \omega_0(\tau)}.\tag{110}$$

Taking the inverse Fourier transform, using

$$\int\frac{d^dq}{(2\pi)^d}e^{i\vec{q}\cdot\vec{r}}e^{-q^2(t_1-t_2)}=\frac{e^{-r^2/[4(t_1-t_2)]}}{[4\pi(t_1-t_2)]^{d/2}},\tag{111}$$

we obtain

$$G^{(0)}_{Mm}(r,t_1t_2)=-i\theta(t_2-t_1)R(t_2,t_1)\frac{e^{-r^2/[4(t_2-t_1)]}}{[4\pi(t_2-t_1)]^{d/2}}.\tag{112}$$

It is straightforward to show explicitly the result we expect from symmetry considerations:

$$G_{mM}^{(0)}(r,t_1t_2) = -i\theta(t_1-t_2)R(t_1,t_2)\frac{e^{-r^2/[4(t_1-t_2)]}}{[4\pi(t_1-t_2)]^{d/2}}. \tag{113}$$

Let us turn our attention to the correlation function. It is useful to introduce the inverse propagators

$$G_{Mm}^{0,-1}(12) = -i[\tilde{\Lambda}(1)+\omega_0(1)]\delta(12), \tag{114}$$

$$G_{mM}^{0,-1}(12) = i[\Lambda(1)-\omega_0(1)]\delta(12), \tag{115}$$

which allows one to write the equation for the correlation function in the form

$$\int d\bar{1}\, G_{mM}^{0,-1}(1\bar{1})G_2^{(0)}(\bar{1}2) = -\int d\bar{1}\Pi_0(1\bar{1})G_{Mm}^{(0)}(\bar{1}2). \tag{116}$$

Multiplying from the left by $G_{mM}^{(0)}$ and using the definition of the inverse gives the symmetric form

$$G_2^{(0)}(12) = -\int d\bar{1}\int d\bar{2}\,G_{mM}^{(0)}(1\bar{1})G_{mM}^{(0)}(2\bar{2})\Pi_0(\bar{1}\,\bar{2}). \tag{117}$$

Taking the Fourier transform and inserting the results for the propagators and Π_0, we obtain

$$G_2^{(0)}(q,t_1t_2) = \theta(t_1-t_0)\,\theta(t_2-t_0)e^{-q^2(t_1+t_2-2t_0)} \times R(t_1,t_0)R(t_2,t_0)\tilde{g}(q), \tag{118}$$

where $\tilde{g}(q)$ is the Fourier transform of the initial correlation function. Henceforth we will suppress writing the step functions in the correlation function. Inverting the Fourier transform, we obtain

$$G_2^{(0)}(r,t_1t_2) = R(t_1,t_0)R(t_2,t_0)\int \frac{d^dq}{(2\pi)^d}e^{i\vec{q}\cdot\vec{r}}\tilde{g}(q)e^{-2q^2T}, \tag{119}$$

where it is convenient to introduce

$$T = \frac{t_1+t_2}{2} - t_0. \tag{120}$$

While we are primarily interested in the long-time scaling properties of our system, we can retain some control over the influence of initial conditions and still be able to carry out the analysis analytically if we introduce the initial condition

$$\tilde{g}(q) = g_0 e^{-(1/2)(q\ell)^2} \tag{121}$$

or

$$g(\vec{r}) = g_0\frac{e^{-(1/2)(r/\ell)^2}}{(2\pi\ell^2)^{d/2}}. \tag{122}$$

Inserting this form into Eq. (119), and doing the wave-number integration, we obtain

$$G_2^{(0)}(\vec{r},t_1t_2) = R(t_1,t_0)R(t_2,t_0)\frac{g_0}{[2\pi(\ell^2+4T)]^{d/2}} \times e^{-\frac{1}{2}r^2/(\ell^2+4T)}. \tag{123}$$

In the long-time limit this reduces to

$$G_2^{(0)}(r,t_1t_2) = R(t_1,t_0)R(t_2,t_0)g_0\frac{e^{-r^2/8T}}{(8\pi T)^{d/2}}. \tag{124}$$

Let us turn now to the quantity $R(t_1,t_2)$ defined by Eq. (110). We have assumed that $\xi(t)\approx 1/L(t)$ and we have $\phi_0 \approx 1/L(t)$, so for long times we can write

$$\omega_0(t) = \xi(t)\psi_0\phi_0(t) = \frac{\omega}{t_c+t}, \tag{125}$$

where ω is a constant we will determine, and t_c is a short-time cutoff which depends on details of the earlier-time evolution. Evaluating the integral

$$\int_{t_2}^{t_1}d\tau\,\omega_0(\tau) = \int_{t_2}^{t_1}d\tau\frac{\omega}{t_c+\tau} = \omega\,\ln\left(\frac{t_1+t_c}{t_2+t_c}\right), \tag{126}$$

we obtain

$$R(t_1,t_2) = \left(\frac{t_1+t_c}{t_2+t_c}\right)^{\omega}. \tag{127}$$

Inserting this result back into Eq. (124) leads to the expression for the correlation function

$$G_2^{(0)}(r,t_1t_2) = g(0)\left(\frac{t_1+t_c}{t_0+t_c}\right)^{\omega}\left(\frac{t_2+t_c}{t_0+t_c}\right)^{\omega}\frac{e^{-r^2/8T}}{(8\pi T)^{d/2}}. \tag{128}$$

If we are to have a self-consistent scaling equation, then the autocorrelation function $(r=0)$, at large equal times $t_1=t_2 = t$, given by

$$S_2^{(0)}(t) = g_0\left(\frac{t}{t_0+t_c}\right)^{2\omega}\frac{1}{(8\pi t)^{d/2}} = t^{2\omega-d/2}\frac{1}{(t_0+t_c)^{2\omega}}\frac{g_0}{(8\pi)^{d/2}}, \tag{129}$$

must have the form $S_2^{(0)}(t) = A_0 t$. Comparing we see the exponent ω must be given by

$$\omega = \frac{1}{2}\left(1+\frac{d}{2}\right), \tag{130}$$

and the amplitude by

$$A_0 = \frac{1}{(t_0+t_c)^{2\omega}}\frac{g_0}{(8\pi)^{d/2}}. \tag{131}$$

One question which arises is whether the time dependence in $\omega_0(t)$ should be viewed as externally driven or self-consistently developed. Rephrasing this question: Does the time evolution for $m(t)$ evolve out of the early-time instability? It is instructive to see that this follows in a natural fashion. Suppose instead of Eq. (125) we assume

$$\omega_0(t)=\frac{\bar{\omega}}{S_2(t)}, \tag{132}$$

which is consistent with

$$\psi_0\xi(t)=\frac{\omega_0(t)}{\phi_0(t)}=\bar{\omega}\sqrt{\frac{\pi}{2S_2(t)}} \tag{133}$$

since $\phi_0(t)=\sqrt{2/\pi S_2(t)}$. Inserting this result for $\omega_0(t)$ into Eq. (110) and Eq. (119) with $r=0$, one finds, at equal times $t_1=t_2=t$, a self-consistent equation for the zeroth-order quantity $S_2^{(0)}(t)$:

$$S_2^{(0)}(t)=\exp\left(2\bar{\omega}\int_{t_0}^{t}\frac{d\tau}{S_2^{(0)}(\tau)}\right)\frac{S_2^{(0)}(t_0)}{[1+\alpha(t-t_0)]^{d/2}}, \tag{134}$$

where

$$S_2^{(0)}(t_0)=\frac{g_0}{(2\pi\ell^2)^{d/2}} \tag{135}$$

and

$$\alpha\equiv 4/\ell^2. \tag{136}$$

This equation is solved in the Appendix, with the result

$$S_2^{(0)}(t)=\left(S_2^{(0)}(t_0)-\frac{2\bar{\omega}}{\alpha(1+d/2)}\right)[1+\alpha(t-t_0)]^{-d/2}+\frac{2\bar{\omega}}{\alpha(1+d/2)}[1+\alpha(t-t_0)], \tag{137}$$

and $\bar{\omega}$ is still undetermined. There are several interesting results which follow. If we look at the long-time limit of $\omega_0(t)$, we obtain

$$\omega_0^{-1}(t)=\frac{S_2^{(0)}(t)}{\bar{\omega}}=\frac{2}{\alpha(1+d/2)}[1+\alpha(t-t_0)]. \tag{138}$$

If we compare with Eq. (125), we regain Eq. (130), and find

$$t_c=\frac{1}{\alpha}-t_0. \tag{139}$$

Thus our procedure is consistent, and the system governed by Eqs. (118) and (132) is driven to grow as desired. Since $S_2^{(0)}(t)$ is now a known function of time we can use Eq. (134) to evaluate $R(t,t_0)$, and obtain the complete solution to this zeroth-order problem over the entire time regime

$$G_2^{(0)}(\vec{r},t_1t_2)=\frac{g_0}{[2\pi(\ell^2+4T)]^{d/2}}e^{-(1/2)r^2/(\ell^2+4T)}\times\left[\frac{S_2^{(0)}(t_1)}{S_2^{(0)}(t_0)}\frac{S_2^{(0)}(t_2)}{S_2^{(0)}(t_0)}[1+\alpha(t_1-t_0)]^{d/2}\times[1+\alpha(t_2-t_0)]^{d/2}\right]^{1/2}.$$

In the long-time limit this can be written, using definition (139) for t_c, in the form

$$G_2^{(0)}(r,t_1t_2)=A_0[(t_1+t_c)(t_2+t_c)]^{(1/2)(1+d/2)}\frac{e^{-r^2/8T}}{T^{d/2}}, \tag{140}$$

with

$$A_0=\frac{2\bar{\omega}}{1+d/2}. \tag{141}$$

Notice that this result for A_0 differs from the result given by Eq. (131), and does not depend on the initial conditions. This is evidence that the coefficient A_0 is nonuniversal. Note, if we choose to enforce the condition $\langle(\nabla m)^2\rangle=1$ at late times [see Eq. (25)] we can fix the parameter $\bar{\omega}$ at this order, with the result

$$\bar{\omega}=\frac{2}{d}\left(1+\frac{d}{2}\right). \tag{142}$$

The general expression for the correlation function can be rewritten in the convenient form

$$G_2^{(0)}(r,t_1t_2)=\sqrt{S_2^{(0)}(t_1)S_2^{(0)}(t_2)}\,\Phi_0(t_1t_2)e^{-(1/2)r^2/(\ell^2+4T)} \tag{143}$$

where, using Eq. (139) for $1/\alpha$,

$$\Phi_0(t_1t_2)=\left(\frac{\sqrt{(t_1+t_c)(t_2+t_c)}}{T+t_c+t_0}\right)^{d/2}. \tag{144}$$

The nonequilibrium exponent is defined in the long-time limit by

$$\frac{G_2^{(0)}(0,t_1t_2)}{\sqrt{S_2^{(0)}(t_1)S_2^{(0)}(t_2)}}=\left(\frac{\sqrt{(t_1+t_c)(t_2+t_c)}}{T+t_0+t_c}\right)^{\lambda_m}, \tag{145}$$

and we obtain the OJK result

$$\lambda_m=\frac{d}{2}. \tag{146}$$

We put the subscript m on λ to indicate the exponent associated with $G_2(12)$. In general $G_2(12)$ and the order parameter correlation function $C(12)$ need not share the same exponents.

Looking at equal times, we have that

$$f_0(x)=\frac{G_2^{(0)}(r,tt)}{S_2^{(0)}(t)}=e^{-x^2/2}, \tag{147}$$

where the scaled length is defined by $\vec{x}=\vec{r}/4t$. $f_0(x)$ is just the well known OJK result for the scaled auxiliary correlation function.

VII. PERTURBATION THEORY FOR FOUR-POINT CUMULANTS

In order to see how things go at higher order in perturbation theory, it is useful to look at the lowest nonzero approximation for the four-point cumulants which enters into the $O(2)$ correction for the two-point cumulant. In order to compute $G_4(1234)$ and $G_{Mmmm}(1234)$ to $O(1)$ in perturbation theory, we see from Eqs. (61) and (62) that we must evaluate

$$Q_4(1234)=\frac{\delta^3}{\delta h(4)\,\delta h(3)\,\delta h(2)}\,\xi(1)\psi_0 \times\int dx\ \mathrm{sgn}(x)P_h(x,1),$$

and

$$\begin{aligned}\hat{Q}_4(1234)&=\frac{\delta^2}{\delta h(4)\,\delta h(3)}\hat{Q}_2(12)\\&=\frac{\delta^2}{\delta h(4)\,\delta h(3)}\int dx\ 2\xi(1)\,\delta(x)\left[G_{Mm}(12)+\langle M(1)\rangle_h\frac{\delta}{\delta h(2)}+\frac{\delta^2}{\delta h(2)\,\delta H(1)}\right]P_h(x,1)\\&=\int dx\ 2\xi(1)\,\delta(x)\left[G_{Mmmm}(1234)P_h(x,1)+G_{Mm}(12)\frac{\delta^2}{\delta h(4)\,\delta h(3)}P_h(x,1)\right.\\&\quad\left.+G_{Mm}(13)\frac{\delta^2}{\delta h(2)\,\delta h(4)}P_h(x,1)+G_{Mm}(14)\frac{\delta^2}{\delta h(2)\,\delta h(3)}P_h(x,1)+\frac{\delta^4}{\delta h(4)\,\delta h(3)\,\delta h(2)\,\delta H(1)}P_h(x,1)\right]\end{aligned} \tag{148}$$

to $O(1)$. Assuming, as we will show self-consistently, that $G_n\approx O[(n/2)-1]$ and $\mathcal{V}_n\approx O(n)$, one easily finds that the $O(1)$ contributions to Q_4 and $\hat{Q}_4$ are given by

$$Q_4^{(1)}(1234)=\omega_0(1)G_4(1234)-\omega_1(1)G_2(14)G_2(13)G_2(12) \tag{149}$$

and

$$\begin{aligned}\hat{Q}_4^{(1)}(1234)=\omega_0(1)G_{Mmmm}(1234)-\omega_1(1)[&G_{Mm}(12)G_2(13)G_2(14)+G_{Mm}(13)G_2(12)G_2(14)+G_{Mm}(14)G_2(12)G_2(13)\\&+G_2(14)G_{Mmmm}(1123)+G_2(13)G_{Mmmm}(1124)+G_2(12)G_{Mmmm}(1134)],\end{aligned} \tag{150}$$

where we have introduced the notation

$$\omega_p(1)=\xi(1)\psi_0\phi_p(1). \tag{151}$$

For future reference it is not difficult to work out the $O(2)$ contributions to Q_4 and $\hat{Q}_4$ given by

$$Q_4^{(2)}(1234)=-\frac{\omega_1(1)}{2}[G_4(1134)G_2(12)+G_4(1124)G_2(13)+G_4(1123)G_2(14)] \tag{152}$$

and

$$\begin{aligned}\hat{Q}_4^{(2)}(1234)=-\frac{\omega_1(1)}{2}&[G_4(1134)G_{Mm}(12)+G_4(1124)G_{Mm}(13)+G_4(1123)G_{Mm}(14)]\\&-\omega_1(1)[G_{Mmmmm}(1134)G_2(12)+G_{Mmmmm}(1124)G_2(13)+G_{Mmmmm}(1123)G_2(14)].\end{aligned} \tag{153}$$

Using Eqs. (150) and (61), we easily find the determining equation for G_{Mmmm} at lowest nontrivial order:

$$\int d\bar{1}\, G_{Mm}^{-1,0}(1\bar{1}) G_{Mmmm}(\bar{1}234) + i\omega_1(1)[G_{Mm}(12)G_2(13)G_2(14) + G_{Mm}(13)G_2(12)G_2(14) + G_{Mm}(14)G_2(12)G_2(13)] = 0. \quad (154)$$

This equation is easily integrated to give

$$G_{Mmmm}(1234) = \int d\bar{1}\, G_{Mm}^{(0)}(1\bar{1})[-i\omega_1(\bar{1})] \times [G_{Mm}(\bar{1}2)G_2(\bar{1}3)G_2(\bar{1}4) + G_{Mm}(\bar{1}3)G_2(\bar{1}2)G_2(\bar{1}4) + G_{Mm}(\bar{1}4)G_2(\bar{1}2)G_2(\bar{1}3)]. \quad (155)$$

Inserting Eq. (149) into Eq. (62), we have

$$G_{mM}^{-1,0}(1\bar{1})G_4(\bar{1}234) + i\omega_1(1)G_2(12)G_2(13)G_2(14) = -\Pi_0(1\bar{1})G_{Mmmm}(\bar{1}234), \quad (156)$$

where an integration over repeated barred indices here and below is assumed. This can be integrated up to give

$$G_4(1234) = -G_{mM}^{(0)}(1\bar{1})\Pi_0(\bar{1}\,\bar{2})G_{Mmmm}(\bar{2}234) + G_{mM}^{(0)}(1\bar{1})(-i\omega_1(\bar{1}))G_2(\bar{1}2)G_2(\bar{1}3)G_2(\bar{1}4). \quad (157)$$

Inserting the result for G_{Mmmm} given by Eq. (155) and using Eq. (117), gives the result

$$G_4(1234) = G_2^{(0)}(1\bar{1})[-i\omega_1(\bar{1})] \times [G_{Mm}(\bar{1}2)G_2(\bar{1}3)G_2(\bar{1}4) + G_{Mm}(\bar{1}3)G_2(\bar{1}2)G_2(\bar{1}4) + G_{Mm}(\bar{1}4)G_2(\bar{1}2)G_2(\bar{1}3)] + G_{Mm}^{(0)}(\bar{1}1)(-i\omega_1(\bar{1}))G_2(\bar{1}2)G_2(\bar{1}3)G_2(\bar{1}4). \quad (158)$$

If we evaluate all of the two-point correlations in G_4 and G_{Mmmm} at lowest order, we obtain our lowest-order approximation for the four-point quantities. Notice that G_4 is properly symmetric under interchange of any two of its labels.

VIII. STRUCTURE OF PERTURBATION THEORY AT HIGHER ORDER

In order to understand the structure of the perturbation theory, one can rather easily see that we should consider two classes of contributions to

$$Q_n(12\ldots n) = \frac{\delta^{n-1}}{\delta h(n)\,\delta h(n-1)\cdots\delta h(2)} \times\left[\int dx\,\xi(1)\sigma_1(x)\int\frac{dk}{2\pi}e^{-ikx}\Phi(k,h,1)\right]. \quad (159)$$

The first class corresponds to all $2n-1$ derivatives with respect to h acting on $\Phi(k,h,1)$, to give, in the zero external field limit, a product of two-point correlation functions of the form

$$Q_{2n}(12,\ldots,2n) = \xi(1)\psi_0\mathcal{V}_{n-1}(1)G_2(12)G_2(13),\ldots G_2(1,2n).$$

There is another set of contributions where all of the derivatives except the first acts on the factor multiplying $\Phi(k,h,1)$, and gives a contribution

$$Q_{2n}(12,\ldots,2n) = \xi(1)\psi_0\mathcal{V}_0(1)G_{2n}(123,\ldots,2n).$$

Since each of these terms is of the same order in ϕ_p, we easily find the proposed result $G_{2p}\approx\phi_{p-1}$. A very similar analysis follows for the case of $\hat{Q}_{2n}$.

IX. TWO-POINT CORRELATION FUNCTION AT HIGHER ORDER

Following the same procedures, it is easy to find that the next order contribution to Q_2, and $\hat{Q}_2$ is of $O(2)$ and given by

$$Q_2^{(2)}(12) = -\frac{\omega_1(1)}{3!}G_4(1112), \quad (160)$$

$$\hat{Q}_2^{(2)}(12) = -\frac{\omega_1(1)}{2}G_{Mmmm}(1112). \quad (161)$$

Inserting Eqs. (104) and (161) into Eq. (65) gives the equation for the response function to second order:

$$G_{Mm}^{-1,0}(1\bar{1})G_{Mm}(\bar{1}2) + i\frac{\omega_1(1)}{2}G_{Mmmm}(1112) = \delta(12).$$

Similarly, the equation determining the correlation function to second order is given by

$$G_{mM}^{-1,0}(1\bar{1})G_2(\bar{1}2) + i\frac{\omega_1(1)}{3!}G_4(1112) = -\Pi_0(1\bar{1})G_{Mm}(\bar{1}2).$$

These equations can be integrated to give

$$G_{Mm}(12) = G_{Mm}^{(0)}(12) + G_{Mm}^{(0)}(1\bar{1})\frac{-i\omega_1(\bar{1})}{2} \times G_{Mmmm}(\overline{111}2) \quad (162)$$

and

$$G_2(12) = -G^{(0)}_{mM}(1\bar{1})\Pi_0(\bar{1}\,\bar{2})G_{Mm}(\bar{2}2) + G^{(0)}_{mM}(1\bar{1})\frac{-i\omega_{\bar{1}}(1)}{3!}G_4(\overline{1112}). \quad (163)$$

Inserting Eq. (162) for G_{Mm} into Eq. (163) gives

$$G_2(12) = G_2^{(0)}(12) + G_2^{(0)}(1\bar{1})\frac{-i\omega_1(\bar{1})}{2}G_{Mmmm}(\overline{1112}) + G^{(0)}_{mM}(1\bar{1})\frac{-i\omega_1(\bar{1})}{3!}G_4(\overline{1112}). \quad (164)$$

Using our $O(1)$ result for G_{Mmmm} gives the $O(2)$ result for the response function

$$G_{Mm}(12) = G^{(0)}_{Mm}(12) + G^{(0)}_{Mm}(1\bar{1})\Sigma^{(2)}_{Mm}(\bar{1}\,\bar{2})G^{(0)}_{Mm}(\bar{2}2), \quad (165)$$

where the lowest-order self-energy contribution is given by

$$\Sigma^{(2)}_{Mm}(12) = \tfrac{1}{2}[-i\omega_1(1)]G^{(0)}_{Mm}(12)G_2^{(0)}(12)^2[-i\omega_1(2)]. \quad (166)$$

Using the results for G_{Mmmm} and G_4 at $O(1)$ leads to the $O(2)$ result for the correlation function

$$G_2(12) = G_2^{(0)}(12) + G_2^{(2,1)}(12) + G_2^{(2,1)}(21) + G_2^{(2,2)}(12),$$

where

$$G_2^{(2,1)}(12) = G_2^{(0)}(1\bar{1})\Sigma^{(2)}_{Mm}(\overline{12})G^{(0)}_{Mm}(\bar{2}2) \quad (167)$$

and

$$G_2^{(2,2)}(12) = -G^{(0)}_{mM}(1\bar{1})\Pi^{(2)}(\overline{12})G^{(0)}_{Mm}(\bar{2}2). \quad (168)$$

The self-energy is the same as for the response function and

$$\Pi^{(2)}(12) = -\frac{1}{3!}[-i\omega_1(1)]G_2^{(0)}(12)^3[-i\omega_1(2)]. \quad (169)$$

We need to evaluate $G_2^{(2,1)}$ and $G_2^{(2,2)}$. The integrations over space, in d dimensions, are straightforward since they involve products of displaced Gaussians. After rescaling the internal time integrations $\bar{t}_1 = Ty_1$, $\bar{t}_2 = Ty_2$, $T = \tfrac{1}{2}(t_1 + t_2)$, we obtain

$$G_2^{(2,1)}(12) = \sqrt{S_0(1)S_0(2)}\,2^{d-1} \times \omega^2\Phi_0(t_1t_2)J_1(x,t_1/T,T),$$

$$J_1(x,t_1/T,T) = \int_{t_0/T}^{t_1/T}dy_1\int_{t_0/T}^{y_1}dy_2R_1(y_1,y_2) \times e^{-(1/2)g_1(y_1,y_2)x^2},$$

$$R_1(y_1,y_2) = \frac{y_1^{d/2-1}y_2^{d/2-1}}{[(y_1+y_2)(3y_1-y_2-(y_1-y_2)^2)]^{d/2}},$$

$$G_2^{(2,2)}(12) = \sqrt{S_0(1)S_0(2)}\,\frac{2^{d-1}}{3}\omega^2\Phi_0(t_1t_2),J_2(x,t_1,t_2),$$

$$J_2(x,t_1,t_2) = \int_{t_0/T}^{t_1/T}dy_1\int_{t_0/T}^{t_2/T}dy_2R_2(y_1,y_2) \times e^{-(1/2)g_2(y_1,y_2)x^2},$$

$$R_2(y_1,y_2) = \frac{y_1^{d/2-1}y_2^{d/2-1}}{[(y_1+y_2)^2(3-y_1-y_2)]^{d/2}},$$

where we have chosen $x^2 = r^2/4T$ and

$$g_1(y_1,y_2) = \frac{3y_1-y_2}{3y_1-y_2-(y_1-y_2)^2}$$

and

$$g_2(y_1,y_2) = \frac{3}{3-y_1-y_2}.$$

The first thing we should do with this result is look at the contribution at this order to the onsite equal-time $t_1 = t_2 = t$ correlation function given by

$$S_2(t) = S_2^{(0)}(t) + 2G_2^{(2,1)}(11) + G_2^{(2,2)}(11) = S_2^{(0)}(t)\left[1 + 2^d\omega^2J_1(0,1,t) + \frac{2^d}{6}\omega^2J_2(0,t,t)\right]$$

where we have the integrals

$$J_1(0,1,t) = \int_{t_0/t}^{1}dy_1\int_{t_0/t}^{y_1}dy_2R_1(y_1,y_2),$$

$$J_2(0,t,t) = \int_{t_0/t}^{1}dy_1\int_{t_0/t}^{1}dy_2R_2(y_1,y_2).$$

The key point here is that $J_1(0,1,t)$ and $J_2(0,t,t)$ are logarithmically divergent at $t\to\infty$. One can show to logarithmic order that

$$J_1(0,1,t) = K_d\ln(t/t_0) + \cdots,$$

where

$$K_d = \int_0^1 dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}},$$

and

$$J_2(0,t,t)=\frac{2}{3^{d/2}}M_d\ln(t/t_0)+\cdots,$$

where

$$M_d=\int_0^1 dz\frac{z^{d/2-1}}{[1+z]^d}=\frac{1}{2}\frac{\Gamma^2(d/2)}{\Gamma(d)}.$$

We have then that

$$S_2(1)=S_2^{(0)}(t)\left[1+\omega^2 2^d\left(K_d+\frac{M_d}{3^{d/2+1}}\right)\ln(t/t_0)+\cdots\right],$$

and a simple exponentiation of this result gives

$$S_2(1)=S_2^{(0)}(t)\left(\frac{t}{t_0}\right)^{\omega^2 2^d[K_d+(M_d/3^{d/2+1}))}(1+\cdots).$$

For self-consistency we must determine ω at this order. Remembering that at lowest order

$$S_2^{(0)}(t)=A_0 t^{2\omega-d/2},$$

we require that

$$S_2(1)=A_0\left(\frac{t}{t_0}\right)^{2\omega-d/2+\omega^2 2^d[K_d+(M_d/3^{d/2+1})]}(1+\cdots)=At$$

which determines ω at this order:

$$2\omega+\omega^2 2^d\left(K_d+\frac{M_d}{3^{d/2+1}}\right)=1+\frac{d}{2}. \tag{170}$$

Then, for example, for $d=2$

$$K_2=\tfrac{1}{4}\ln 3,\quad M_2=\tfrac{1}{2}$$

and

$$\omega=\frac{\sqrt{1+2\ln 3+4/9}-1}{\ln 3+2/9}=0.687687370\ldots.$$

For large d analytical progress leads to the results

$$K_d\approx\frac{2}{d}\frac{1}{2^d}+\cdots,$$

$$M_d\approx\sqrt{\frac{2\pi}{d}}\frac{1}{2^d}+\cdots,$$

and

$$\omega=\frac{d}{2}(\sqrt{2}-1).$$

A numerical determination of ω shows that it is approximately linear with d over the whole range of d.

The $t/t_0\to\infty$ singularity in $G_2(12)$ at $O(2)$ can be regulated by turning our attention from $G_2(12)$ to the quantities $\Phi(t_1t_2)$ and $f(x,t_1/t_2)$ defined by

$$G_2(12)=\sqrt{S(1)S(2)}\Phi(t_1t_2)f(x,t_1/t_2) \tag{171}$$

and the constraint $f(x=0,t_1/t_2)=1$. If we write

$$G_2(12)=G_2^{(0)}(12)+\Delta G_2(12)$$

and

$$S_2(1)=S_2^{(0)}(1)+\Delta S_2(1),$$

then

$$\begin{aligned}\Phi(t_1t_2)f(x,t_1/t_2)&=\frac{G_2(12)}{\sqrt{S_2(1)S_2(2)}}\\&=\Phi^{(0)}(t_1,t_2)f_0(x)\\&\quad\times\left(1-\frac{1}{2}\left[\frac{\Delta S_2(1)}{S_2^{(0)}(1)}+\frac{\Delta S_2(2)}{S_2^{(0)}(2)}\right]\right)\\&\quad+\frac{\Delta G_2(12)}{\sqrt{S_2^{(0)}(1)S_2^{(0)}(2)}}.\end{aligned} \tag{172}$$

We can then separate the contribution to the on-site correlation function $\Phi(t_1,t_2)$ from the general x dependence and write

$$\Phi(t_1,t_2)=\Phi_0(t_1,t_2)[1+\omega^2 2^d\Delta\Phi(t_1,t_2)], \tag{173}$$

$$f(x,t_1/t_2)=f_0(x)[1+\omega^2 2^d W(x,t_1/t_2)], \tag{174}$$

where

$$\Delta\Phi=\tfrac{1}{2}\Phi_1+\tfrac{1}{6}\Phi_2,$$

$$\begin{aligned}\Phi_1=&\tfrac{1}{2}\int_{t_0/T}^{t_1/T}dy_1\int_{t_0/T}^{y_1}dy_2R_1(y_1,y_2)\\&+\tfrac{1}{2}\int_{t_0/T}^{t_2/T}dy_1\int_{t_0/T}^{y_1}dy_2R_1(y_1,y_2)\\&-\tfrac{1}{2}\int_{t_0/t_1}^{1}dy_1\int_{t_0/t_1}^{y_1}dy_2R_1(y_1,y_2)\\&-\tfrac{1}{2}\int_{t_0/t_2}^{1}dy_1\int_{t_0/t_2}^{y_1}dy_2R_1(y_1,y_2),\end{aligned} \tag{175}$$

$$\Phi_2=\tfrac{1}{6}\int_{t_0/T}^{t_1/T}dy_1\int_{t_0/T}^{t_2/T}dy_2R_2(y_1,y_2)-\tfrac{1}{12}\int_{t_0/t_1}^{1}dy_1\int_{t_0/t_1}^{1}dy_2R_2(y_1,y_2)-\tfrac{1}{12}\int_{t_0/t_2}^{1}dy_1\int_{t_0/t_2}^{1}dy_2R_2(y_1,y_2)$$

and

$$W(x,t_1/t_2)=\tfrac{1}{2}\int_0^{t_1/T}dy_1\int_0^{y_1}dy_2R_1(y_1,y_2)[e^{-(1/2)\Delta g_1x^2}-1]+\tfrac{1}{2}\int_0^{t_2/T}dy_1\int_0^{y_1}dy_2R_1(y_1,y_2)[e^{-(1/2)\Delta g_1x^2}-1]$$
$$+\tfrac{1}{6}\int_0^{t_1/T}dy_1\int_0^{t_2/T}dy_2R_2(y_1,y_2)[e^{-(1/2)\Delta g_2x^2}-1],$$

where

$$\Delta g_1=g_1(y_1,y_2)-1=\frac{(y_1-y_2)^2}{3y_1-y_2-(y_1-y_2)^2}$$

and

$$\Delta g_2=g_2(y_1,y_2)-1=\frac{y_1+y_2}{3-y_1-y_2}.$$

Let us first look first at $\Delta\Phi(t_1,t_2)$ and the integrals Φ_1 and Φ_2. An investigation of Φ_1 shows that it is a regular quantity for $t_1\gg t_2$ or $t_2\gg t_1$. Turning to Φ_2, however, one finds that it is logarithmically divergent in these limits:

$$\Phi_2=\frac{2M_d}{3^{d/2}}\ln\left(\frac{\sqrt{t_1t_2}}{T}\right)+\cdots\ .$$

This means that to second order we have

$$\Phi(t_1,t_2)=\left(\frac{\sqrt{t_1t_2}}{T}\right)^{d/2}\left[1+\omega^2\frac{2^dM_d}{3^{d/2}+1}\ln\left(\frac{\sqrt{t_1t_2}}{T}\right)\right],$$

which can be exponentiated to give

$$\Phi(t_1,t_2)=\left(\frac{\sqrt{t_1t_2}}{T}\right)^{\lambda_m}(1+\cdots),\tag{176}$$

where

$$\lambda_m=\frac{d}{2}+\omega^2\frac{2^dM_d}{3^{d/2+1}}.\tag{177}$$

It is just this expression for λ discussed in Sec. II.

Turning next to $f(x,t_1/t_2)$, one of the first things to note is that it is for small x. Looking at the first term in the power series expansion in x^2, we obtain

$$W(x,t_1/t_2)=-\tfrac{1}{2}x^2[I\tfrac{1}{2}I_1^{(2)}(t_1/T)+\tfrac{1}{2}I_1^{(2)}(t_2/T)$$
$$+\tfrac{1}{6}I_2^{(2)}(t_1/t_2)],$$

where

$$I_1^{(2)}(t_1/T)=\int_0^{t_1/T}dy_1\int_0^{y_1}dy_2R_1(y_1,y_2)\Delta g_1(y_1,y_2),$$

$$I_2^{(2)}(t_1/t_2)=\int_0^{t_1/T}dy_1\int_0^{t_2/T}dy_2R_2(y_1,y_2)\Delta g_2(y_1,y_2),$$

where since $T\gg t_0$, the lower limits can be set to zero. After making the change of variables $y_2=y_1z$ one finds that one can perform the integral over y_1 to obtain

$$I_1^{(2)}(t_1/T)=\frac{2}{d}[J_d(t_1/T)-K_d(t_1/T)],\tag{178}$$

where

$$J_d(t_1/T)=\int_0^{t_1/T}dz\frac{z^{d/2-1}}{[(1+z)^2(2-z)]^{d/2}},\tag{179}$$

$$K_d(t_1/T)=\int_0^{t_1/T}dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}},\tag{180}$$

$$I_2^{(2)}(t_1/T)=\frac{2}{d}\left[L_d(t_1/t_2)+L_d(t_1/t_2)-\frac{2}{3^{d/2}}M_d\right],\tag{181}$$

and

$$L_d(t_1/t_2)=\int_0^{t_1/t_2}dz\frac{z^{d/2-1}}{\left[(1+z)^2\left(3-\dfrac{t_2}{T}(1+z)\right)\right]^{d/2}}.\tag{182}$$

Notice that at equal times, $K_d(1)=K_d$, $L_d(1)=J_d(1)$, and all of these integrals are well behaved.

There is one last regularization that must be carried out before the perturbation theory expression for the correlation function can be used for all values of the parameters. Consider that W can be written as the sum of three terms:

$$W^{(1)}(x,t_1/t_2)=\tfrac{1}{2}\int_0^{t_1/T}dy_1\int_0^{y_1}dy_2R_1(y_1,y_2)$$
$$\times[e^{-(1/2)\Delta g_1x^2}-1],$$

$$W^{(1)}(x,t_2/t_1)=\tfrac{1}{2}\int_0^{t_2/T}dy_1\int_0^{y_1}dy_2R_1(y_1,y_2)\times[e^{-(1/2)\,\Delta g_1x^2}-1],$$

$$W^{(2)}(x,t_1/t_2)=\tfrac{1}{6}\int_0^{t_1/T}dy_1\int_0^{t_2/T}dy_2R_2(y_1,y_2)\times[e^{-(1/2)\,\Delta g_2x^2}-1].$$

In $W^{(1)}(x,t_1/t_2)$ let $y_2=y_1z$, which leads to the result

$$W^{(1)}(x,t_1/t_2)=\tfrac{1}{2}\int_0^{t_1/T}\frac{dy_1}{y_1}\times\int_0^1dz\frac{z^{d/2-1}}{\{(1+z)[3-z-y_1(1-z)^2]\}^{d/2}}\times[\exp^{-\frac{1}{2}x^2\frac{y_1(1-z)^2}{(3-z-y_1(1-z)^2)}}-1]. \tag{183}$$

Notice in this integral that there is an apparent log divergence for small y_1 which is canceled between the exponential term and the subtraction term. Note that the log terms survives if x is large enough. To pick out the log term we can expand about small y_1 except for the contribution x^2y_1 and write

$$W^{(1)}(x,t_1/t_2)=\tfrac{1}{2}\int_0^{t_1/T}\frac{dy_1}{y_1}\int_0^1dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}\times\left[\exp\left(-\tfrac{1}{2}x^2\frac{y_1(1-z)^2}{(3-z)}\right)-1\right]+\Delta W^{(1)}(x,t_1/t_2),$$

where $\Delta W^{(1)}(x,t_1/t_2)$ has no singularity for large x. If we now make the change of variables

$$y_1=\frac{t_1s}{T(1+x^2)}, \tag{184}$$

in the leading integral we obtain

$$W^{(1)}(x,t_1/t_2)=\frac{1}{2}\int_0^{1+x^2}\frac{ds}{s}\int_0^1dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}\times\left[\exp-\frac{1}{2}\frac{x^2}{1+x^2}\frac{t_1}{T}\frac{s(1-z)^2}{(3-z)}-1\right]+\cdots,$$

where the $\cdots$ refer to contributions which are regular. The integral over s can be divided into a regular part from 0 to 1 and the singular part for large x given by

$$W^{(1)}(x,t_1/t_2)=\frac{1}{2}\int_1^{1+x^2}\frac{ds}{s}\int_0^1dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}\times\left[\exp\left(-\tfrac{1}{2}\frac{x^2}{1+x^2}\frac{t_1}{T}\frac{s(1-z)^2}{(3-z)}\right)-1\right]+\cdots,$$

The singular part is now isolated in the second piece that does not have exponential convergence for large s. We have then

$$W^{(1)}(x,t_1/t_2)=-\tfrac{1}{2}K_d\ln(1+x^2)+\cdots. \tag{185}$$

Clearly,

$$W^{(1)}(x,t_2/t_1)=-\tfrac{1}{2}K_d\ln(1+x^2)+\cdots. \tag{186}$$

Turning to the third contribution, we again isolate the small y_1 and y_2 behaviors, and expanding in x^2 in places which do not contribute to the singularity gives

$$W^{(2)}(x,t_1/t_2)=\frac{1}{6}\int_0^{t_1/T}dy_1\int_0^{t_2/T}dy_2\frac{y_1^{d/2-1}y_2^{d/2-1}}{(y_1+y_2)^d}\frac{1}{3^{d/2}}\times[\exp^{-(1/6)x^2(y_1+y_2)}-1]+\cdots.$$

Next make the coordinate transformations

$$y_1=\frac{s_1t_1}{T(1+x^2)}, \tag{187}$$

$$y_2=\frac{s_2t_2}{T(1+x^2)}, \tag{188}$$

which results in the leading contribution to the integral

$$W^{(2)}(x,t_1/t_2)=\frac{1}{2(3)^{d/2+1}}\int_0^{1+x^2}ds_1\int_0^{1+x^2}ds_2\frac{s_1^{d/2-1}s_2^{d/2-1}}{(s_1+s_2)^d}\times\left[\exp\left[-\tfrac{1}{6}\frac{x^2}{1+x^2}\left(\frac{s_1t_1}{T}+\frac{s_2t_2}{T}\right)\right]-1\right]+\cdots.$$

Again, the leading behavior for large x comes from terms which do not have exponential convergence for large s_1 and s_2:

$$W^{(2)}(x,t_1/t_2)=-\frac{1}{23^{d/2+1}}\int_0^{1+x^2}ds_1\times\int_0^{1+x^2}ds_2\frac{s_1^{d/2-1}s_2^{d/2-1}}{(s_1+s_2)^d}+\cdots.$$

If we let $s_1\to 1/s_1$ and $s_2\to 1/s_2$ then this integral takes the form

$$W^{(2)}(x,t_1/t_2)=-\frac{1}{23^{d/2+1}}\int_{1/(1+x^2)}^{1}ds_1 \times\int_{1/(1+x^2)}^{1}ds_2\frac{s_1^{d/2-1}s_2^{d/2-1}}{(s_1+s_2)^d}+\cdots.$$

This integral was evaluated in our treatment of Φ with the result

$$W^{(2)}(x,t_1/t_2)=-\frac{1}{23^{d/2+1}}2M_d\ln\left[\frac{1+x^2}{2}\right]+\cdots,$$

Then to leading order for large x we have

$$W(x,t_1/t_2)=-K_d\ln(1+x^2)-\frac{1}{3^{d/2+1}}M_d\ln\left[\frac{1+x^2}{2}\right]+\cdots.$$

The scaled correlation function then has the form

$$f(x,t_1/t_2)=f_0(x)\left[1-\omega^2 2^{d+1}\left(K_d+\frac{M_d}{3^{d/2+1}}\right)\ln(1+x^2)\right]\times[1+\cdots]. \tag{189}$$

This can be exponentiated to obtain

$$f(x,t_1/t_2)=\frac{f_0(x)}{(1+x^2)^{\nu_m/2}}[1+\cdots], \tag{190}$$

where the exponent governing the large x behavior is given by

$$\nu_m=\omega^2 2^{d+1}\left(K_d+\frac{M_d}{3^{d/2+1}}\right). \tag{191}$$

X. ORDER PARAMETER CORRELATION FUNCTION

A. Perturbation expansion

We turn next to the connection between the correlation function for the auxiliary field m and the order parameter correlation function. If we look at the problem using the transformation given by Eq. (28), we have

$$C(12)=\langle\psi(1)\psi(2)\rangle =\langle\sigma(1)\sigma(2)\rangle+\langle\sigma(1)u(2)\rangle +\langle u(1)\sigma(2)\rangle+\langle u(1)u(2)\rangle.$$

The key point is that since $u(1)$ vanishes exponentially for large $|m(1)|$, the averages over these fields are down by a factor of L^{-2} relative to the averages over the field $\sigma(1)$. Thus $\langle\sigma(1)\sigma(2)\rangle\approx O(1)$ as $L(t)\to\infty$, while $\langle\sigma(1)u(2)\rangle$ and $\langle u(1)\sigma(2)\rangle$ are of $O(L^{-2})$ and $\langle u(1)u(2)\rangle$ of $O(L^{-4})$. In the scaling regime we have

$$C(12)=\langle\sigma(1)\sigma(2)\rangle. \tag{192}$$

This quantity can be evaluated using the two-point probability distribution

$$P_h(x_1x_2,1,2)=\langle\delta(x_1-m(1))\delta(x_2-m(2))\rangle_h \tag{193}$$

and the correlation functions are obtained at zero external field via

$$C(12)=\int dx_1\int dx_2\sigma(x_1)\sigma(x_2)P_0(x_1x_2,12). \tag{194}$$

More generally, we can treat the set of correlation functions

$$C_{n\ell}(12)=\langle\sigma_n(1)\sigma_\ell(2)\rangle =\int dx_1\int dx_2\sigma_n(x_1)\sigma_\ell(x_2)P_0(x_1x_2,12).$$

Since we have computed the auxiliary-field correlation functions to $O(2)$, we also need to determine $C_{n\ell}(12)$ to second order. This expansion can be developed as follows. As in the case of the one-point quantity $P_h(x_1,1)$, we again use the integral representation for the δ function to obtain

$$P_h(x_1,x_2,12)=\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}e^{-ik_1x_1}e^{-ik_2x_2}\langle e^{\mathcal{H}(12)}\rangle_h$$

where $\mathcal{H}(12)\equiv ik_1m(1)+ik_2m(2)$. The average of the exponential is precisely of the form which can be rewritten in terms of cumulants:

$$\langle e^{\mathcal{H}(12)}\rangle_h=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}G_{\mathcal{H}}^{(n)}(12)\right], \tag{195}$$

where $G_{\mathcal{H}}^{(n)}(12)$ is the nth-order cumulant for the field $\mathcal{H}(12)$. These cumulants can all be expressed in terms of the m-field cumulants. In this section we can work directly in terms of zero external fields, where all odd m-field cumulants vanish. Here we will need

$$G_{\mathcal{H}}^{(2)}(12)=(ik_1)^2G_2(11)+(ik_2)^2G_2(22) +2(ik_1)(ik_2)G_2(12),$$

$$G_{\mathcal{H}}^{(4)}(12)=(ik_1)^4G_4(1111)+4(ik_1)^3(ik_1)G_4(1112) +6(ik_1)^2(ik_2)^2G_4(1122) +4(ik_1)(ik_2)^3G_4(1222)+(ik_2)^4G_4(2222),$$

and

$$G_{\mathcal{H}}^{(6)}(12)=(ik_1)^6G_6(111111) +6(ik_1)^5(ik_2)G_6(111112) +15(ik_1)^4(ik_2)^2G_6(111122) +20(ik_1)^3(ik_2)^3G_6(111222) +15(ik_1)^2(ik_2)^4G_6(112222) +6(ik_1)(ik_2)^5G_6(122222) +(ik_2)^6G_6(222222).$$

Working to second order in the ϕ expansion requires keeping terms

$$\langle e^{\mathcal{H}(12)}\rangle=\Phi_0(k_1,k_2,12)\times\left[1+\frac{1}{4!}G^{(4)}_{\mathcal{H}}+\frac{1}{2}\left(\frac{1}{4!}G^{(4)}_{\mathcal{H}}\right)^2+\frac{1}{6!}G^{(6)}_{\mathcal{H}}\cdots\right] \tag{196}$$

where in zero external field $\Phi_0(k_1,k_2,12)$ is given by

$$\Phi_0(k_1k_2,12)=\exp-\tfrac{1}{2}[k_1^2G_2(11)+k_2^2G_2(22)+2k_1k_2G_2(12)]. \tag{197}$$

B. Terms of $O(1)$

Let us look first at the theory keeping terms up to $O(1)$. It is then straightforward to show that the two-point probability distribution is given by

$$P_0(x_1,x_2;12)=\left[1+\frac{1}{4!}\left(G_4(1111)\frac{d^4}{dx_1^4}+4G_4(1112)\times\frac{d^3}{dx_1^3}\frac{d}{dx_2}+6G_4(1122)\frac{d^2}{dx_1^2}\frac{d^2}{dx_2^2}+4G_4(1222)\frac{d}{dx_1}\frac{d^3}{dx_2^3}+G_4(2222)\frac{d^4}{dx_2^4}\right)\right]\times P_0^{(0)}(x_1,x_2;12), \tag{198}$$

where $P_0^{(0)}(x_1,x_2;12)$ is given by

$$P_0^{(0)}(x_1x_2,12)=\frac{1}{2\pi}\frac{\gamma(12)}{\sqrt{S_2(1)S_2(2)}}\times\exp\left[-\frac{\gamma^2(12)}{2S_2(1)S_2(2)}[x_1^2S_2(2)+x_2^2S_2(1)-2G_2(12)x_1x_2]\right], \tag{199}$$

$$\gamma(12)=\frac{1}{\sqrt{1-f^2(12)}}, \tag{200}$$

and

$$f(12)=\frac{G_2(12)}{\sqrt{S_2(1)S_2(2)}}. \tag{201}$$

Notice that we already have one resummation here since it is the full G_2 which appears in $P_0^{(0)}(x_1x_2,12)$. The general set of two-point order parameter correlation functions are given, up to $O(1)$, by

$$C_{n\ell}(12)=C_{n\ell}^{(0)}(12)+C_{n\ell}^{(1)}(12)\cdots, \tag{202}$$

where the first-order correction to the leading Gaussian behavior is given explicitly by

$$C_{n\ell}^{(1)}(12)=\frac{1}{4!}G_4^{(1)}(1111)C_{n+4,\ell}^{(0)}(12)+\frac{1}{3!}G_4^{(1)}(1112)C_{n+3,\ell+1}^{(0)}(12)+\frac{1}{4}G_4^{(1)}(1122)C_{n+2,\ell+2}^{(0)}(12)+\frac{1}{3!}G_4^{(1)}(1222)C_{n+1,\ell+3}^{(0)}(12)+\frac{1}{4!}G_4^{(1)}(2222)C_{n,\ell+4}^{(0)}(12),$$

where we remember that $C_{n\ell}^{(0)}(12)$ is a known functional of the exact $G_2(12)$, and that the G_4's are also the exact quantities. Thus $C_{n\ell}^{(0)}(12)$ and $C_{n\ell}^{(1)}(12)$ both contain contributions of $O(2)$.

Thus determining the $C_{n\ell}$ to first order requires first determining G_2 to first order, then evaluating the $C_{n\ell}^{(0)}(12)$ as functions of G_2, and finally the evaluation of the various two-point contractions of $G_4(1234)$ evaluated at first order. We have already evaluated $G_2^{(0)}$ and we have the appropriate first-order expression for $G_4(1234)$. Notice that it is the same set of contracted four-point quantities which enter into the determination of any set n,ℓ. Let us restrict our subsequent analysis to the correlation functions $C_{00}(12)$. The first-order correction to the leading Gaussian result

$$C_{00}^{(0)}(12)=\frac{2}{\pi}\psi_0^2\sin^{-1}f(12) \tag{203}$$

is given by

$$C_{00}^{(1)}(12)=\frac{1}{4!}G_4^{(1)}(1111)C_{4,0}^{(0)}(12)+\frac{1}{3!}G_4^{(1)}(1112)C_{3,1}^{(0)}(12)+\frac{1}{4}G_4^{(1)}(1122)C_{2,2}^{(0)}(12)+\frac{1}{3!}G_4^{(1)}(1222)\times C_{1,3}^{(0)}(12)+\frac{1}{4!}G_4^{(1)}(2222)C_{0,4}^{(0)}(12). \tag{204}$$

The zeroth-order correlation functions $C_{n,\ell}^{(0)}(12)$ can be evaluated using the identities

$$C_{n+1,\ell+1}^{(0)}(12)=\frac{\partial}{\partial G_2(12)}C_{n,\ell}^{(0)}(12) \tag{205}$$

and

$$C_{n+2,\ell}^{(0)}(12)=2\frac{\partial}{\partial S_2(1)}C_{n,\ell}^{(0)}(12) \tag{206}$$

derived in the TUG. Starting with the expression for $C_{0,0}^{(0)}(12)$ given by Eq. (203), all of the other quantities can be calculated by taking derivatives. A summary of the results we need for $C_{0,0}^{(1)}(12)$ is given by

$$C_{40}^{(0)}(12)=\frac{2}{\pi}\frac{\psi_0^2}{S_2^2(1)}\gamma^3 f(3-2f^2), \tag{207}$$

$$C_{31}^{(0)}(12)=-\frac{2}{\pi}\frac{\psi_0^2}{S_2^{3/2}(1)S_2^{1/2}(2)}\gamma^3, \tag{208}$$

$$C_{22}^{(0)}(12)=\frac{2}{\pi}\frac{\psi_0^2}{S_2(1)S_2(2)}\gamma^3 f. \tag{209}$$

Next we need the various contracted four-point cumulants appearing in Eq. (204). These can all be evaluated using Eq. (158). First we have the fully contracted four-point cumulant

$$G_4^{(0)}(1111)=-4\omega 2^d S_2^2(1)L_d, \tag{210}$$

where

$$L_d=\int_0^1 dy\,\frac{y^{d/2-1}}{[(2-y)(1+y)^2]^{d/2}} \tag{211}$$

is the equal-time limit of $L_d(t_1/t_2)$ defined by Eq. (182). Next we have

$$G_4^{(0)}(1122)=-2\omega 2^d S_2(1)S_2(2)\times[\xi^{d/2}W_3(x_1,\xi)+\xi^{-d/2}W_3(x_2,\xi^{-1})], \tag{212}$$

where

$$W_3(x_1,\xi)=\int_0^1 dy\,\frac{y^{d/2-1}}{\{(\xi+y)[1+\xi+y(1-y)]\}^{d/2}}e^{-(1/2)x_1^2\tilde{g}_0(\xi,y)},$$

$$\xi=\frac{t_2}{t_1},\quad x_1^2=r^2/(4t_1),$$

and

$$\tilde{g}_0(\xi,y)=\frac{4}{1+\xi+y(1-y)}.$$

Finally we need

$$G_4^{(0)}(1112)=-\omega 2^d S_2(1)S_2(2)\xi^{d/4-1/2}[3W_1(x_1,\xi)+W_2(x_1,\xi)], \tag{213}$$

$$W_1(x_1,\xi)=\int_0^1 dy\; y^{d/2-1}\left[\frac{2}{(1+y)[1+3\xi+y(3-\xi)-2y^2]}\right]^{d/2}e^{-(1/2)x_1^2\tilde{g}_2(\xi,y)}, \tag{214}$$

$$W_2(x_1,\xi)=\int_0^{\xi} dy\; y^{d/2-1}\left[\frac{2}{(1+y)^2(1+3\xi-2y)}\right]^{d/2}e^{-(1/2)x_1^2\tilde{g}_2(\xi,y)},$$

$$\tilde{g}_1(\xi,y)=\frac{6}{1+3\xi-2y},$$

and

$$\tilde{g}_2(\xi,y)=\frac{4(1+\xi)}{[1+3\xi+y(3-\xi)-2y^2]}.$$

Note the check on Eqs. (212) and (213) that they reduce to Eq. (210) as $2\rightarrow 1$. Pulling all of these results together, we have the $O(1)$ corrections to the order-parameter correlation function

$$\begin{aligned}C_{0,0}^{(1)}(12)=\frac{\omega 2^d\gamma^3}{3\pi}\{&-2L_d f(3-2f^2)\\&+\xi^{d/4}[3W_1(x_1,\xi)+W_2(x_1,\xi)]\\&+\xi^{-d/4}[3W_1(x_2,\xi^{-1})+W_2(x_2,\xi^{-1})]\\&-3\xi^{d/2}fW_3(x_1,\xi)-3\xi^{-d/2}fW_3(x_2,\xi^{-1})\}.\end{aligned} \tag{215}$$

There are several limits of interest. First consider the on-site unequal-time $t_1 \gg t_2$ limit, which gives the nonequilibrium exponent λ. In this limit f is small, and has the form for large ξ given by Eq. (145) which, in our notation here, reads

$$f = A_f^{(0)} \xi^{-d/4}, \tag{216}$$

where $A_f^{(0)} = 2^{d/2}$ at lowest order. Similarly the zeroth-order order parameter correlation function has the form

$$C^{(0)}(0,t_1,t_2) = \frac{2}{\pi} \psi_0^2 A_f^{(0)} \xi^{-d/4}. \tag{217}$$

Second-order contributions to f change the exponent λ from $d/2$ to the expression given by Eq. (13). Do the new first-order terms give contributions which change λ at first order? To answer this question we need to set $r=0$ and take ξ large in $C_{0,0}^{(1)}(12)$. The main results we need for large ξ are

$$W_1(0,\xi) = \xi^{-d/2} q_1(d),$$

$$W_2(0,\xi) = \xi^{-d/2} q_0(d),$$

$$W_1(0,\xi^{-1}) = q_2(d),$$

$$W_2(0,\xi^{-1}) = \frac{4}{d} d^{d/2} \xi^{-d/2},$$

$$W_3(0,\xi) = \frac{2}{d} \xi^{-d},$$

$$W_3(0,\xi^{-1}) = [\ln\xi + q_3(d)],$$

with the dimensionality dependent quantities defined by

$$q_0(d) = \left(\frac{2}{3}\right)^{d/2} \frac{\Gamma^2(d/2)}{\Gamma(d)},$$

$$q_1(d) = \int_0^1 dy\ y^{d/2-1} \left[\frac{2}{(1+y)(3-y)}\right]^{d/2} = 2^{d/2} K_d,$$

$$q_2(d) = \int_0^1 dy\ y^{d/2-1} \left[\frac{2}{(1+y)(1+3y-2y^2)}\right]^{d/2}.$$

We see then that these contributions do not contribute terms to the exponent λ at first order. One has only a contribution to the amplitude

$$C^{(1)}(0,t_1,t_2) = \frac{\omega 2^d}{3\pi} \xi^{-d/4} \times [-6 f_0 L_d + 3 q_1(d) + q_0(d) + 3 q_2(d)].$$

For equal times $t_1 = t_2 = t$ and $\xi = 1$, the expression for $C_{00}^{(1)}(12)$ simplifies significantly. All of the W's share the same integrand except for the $\tilde{g}$'s multiplying x^2 in the argument of the exponential. After considerable manipulation we have

$$C_{0,0}^{(1)}(12) = \frac{\omega 2^d \gamma^3}{3\pi} \int_0^1 dy \frac{y^{d/2-1}}{[(2-y)(1+y)^2]^{d/2}} \times [-2e^{-(1/2)x^2}(3-2f^2) + 2e^{-(1/2)x^2 \tilde{g}_1(1,y)} + 6(1-f)e^{-(1/2)x^2 \tilde{g}_0(1,y)}]. \tag{218}$$

For large x one has, since $\tilde{g}_0(1,y) > 1$, $\tilde{g}_1(1,y) > 1$, that only the fully contracted four-point cumulant contributes and

$$C_{0,0}^{(1)}(12) = \frac{\omega 2^d}{3\pi}(-6L_d) f(12), \tag{219}$$

which does not give a first-order contribution to the exponent ν.

The last point to be discussed in the $O(1)$ evaluation of the order parameter correlation function is that one must be careful about the behavior at short-scaled distances at equal times. Let us write

$$C_{0,0}^{(1)}(12) = \gamma^3 \Delta(x), \tag{220}$$

where $\Delta(x)$ can be read off from Eq. (218). Then for small x we see that $\Delta(x)$ goes to zero as x^2,, while γ^3 blows up as $1/x^3$. Such singularities are unphysical, and indicate that we are expanding about a singular order parameter interfacial profile in this regime. It is just these singularities which give rise to Porod's law. One possible resummation is

$$\mathcal{F}(x) = \frac{2}{\pi} \sin^{-1}\left[f(x) - \frac{\pi}{2} \frac{\Delta(x)}{q_0}\right] - \frac{\pi}{2q_0} \Delta(x) \frac{1}{\sqrt{1 - f(x)^2 + 4q_0/\pi}} \tag{221}$$

and a reasonable choice for q_0 is $q_0^2 = \Delta(\bar{x})$, where $\bar{x}$ is some renormalization point chosen such that the scaling function is smooth.

We can then conclude that the first-order correction to the order parameter correlation function do not lead to corrections to the exponents λ and ν. Thus, to $O(1)$, $G_2(12)$ and $C(12)$ share the same OJK exponents.

C. Terms of $O(2)$

Turning to the more involved case of the terms of $O(2)$ contributions to $C(12)$, we find four types of terms. There are the terms in $G_2(12)$ which are of $O(2)$ which must be included in the contribution given by Eq. (203) and which have already been evaluated. Inserting these results for f into Eq. (203) we find, since f is small in the regime controlled by ν and λ, that the order parameter correlation function picks up the same corrections leading to λ_m and ν_m. Let us turn now to the other three contributions.

(i) One has terms of $O(2)$ from Eq. (204) where the leading order $G_4^{(1)}$ contributions are replaced by their next-order corrections $G_4^{(2)}$.

(ii) The leading G_6 contribution in Eq. (196) is of $O(2)$, and contributes to $C^{(2)}(12)$.

(iii) The term $(G_{\mathcal{H}}^{(4)})^2$ in Eq. (196) is also of $O(2)$.

Just as in Eq. (198), the factors of k_i in Eq. (196) just lead to higher-order subscripts in the multiplicative factors of $C^{(0)}_{n,\ell}(12)$. Thus in $C^{(2)}(12)$ there are contributions of the form

$$\int dx_1\sigma(x_1)\int dx_2\sigma(x_2)\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}e^{-ik_1x_1}e^{-ik_2x_2}$$
$$\times\frac{20}{6!}(ik_1)^3(ik_2)^3G_6(111222)$$
$$=C^{(0)}_{3,3}(12)\frac{20}{6!}G_6(111222)$$

and

$$\int dx_1\sigma(x_1)\int dx_2\sigma(x_2)\int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}e^{-ik_1x_1}e^{-ik_2x_2}$$
$$\times\frac{1}{2}\left(\frac{6(ik_1)^2(ik_2)^2}{4!}\right)^2G_4^2(1122)$$
$$=C^{(0)}_{4,4}(12)\frac{36}{2(4!)^2}G_4^2(1122).$$

While there are a great many terms to be analyzed from all three sets of contributions, none appear to give corrections to the exponents λ and ν. Thus it appears to $O(2)$ that $\lambda=\lambda_m$ and $\nu=\nu_m$.

XI. EQUATION OF MOTION CONSIDERATIONS

In Sec. X we determined the order parameter correlation function by relating it to the auxiliary-field correlation function. It is also instructive to consider using equation of motion methods for determining the order parameter scaling function as in Ref. [3]. Let us return to the equation of motion for the order parameter given by Eq. (39). We will limit the discussion for simplicity to the case of equal times where the equation of motion for the order-parameter correlation function can be written

$$\left(\frac{\partial}{\partial t}-2\nabla_R^2\right)C(\vec{R},t)=-2\langle\sigma_2(1)(\nabla m(1))^2\sigma(2)\rangle$$
$$\equiv -2K(12). \tag{222}$$

Assuming we have a scaling solution $C(\vec{R},t)=F(x)$, we easily find that the equation of motion takes the form

$$\vec{x}\cdot\nabla F(x)+\nabla^2F(x)=L^2K(12). \tag{223}$$

We can then focus on $K(12)$. At leading order, it is easy to show that $K(12)$ can be written in terms of the two-point probability distribution as

$$K(12)=\langle(\nabla m(1))^2\rangle C_{20}(12)+\overline{K}^{(0)}(12), \tag{224}$$

where $C_{20}(12)$ can be evaluated using Eq. (203) in Eq. (206):

$$C^{(0)}_{20}(12)=-\frac{2}{\pi(S_2^{(0)}(1))^2}f(x)\gamma(x). \tag{225}$$

At lowest order we can also evaluate

$$\overline{K}^{(0)}(12)=\left[\nabla_i^{(3)}\nabla_i^{(4)}\int dx_1\int dx_2\sigma_2(x_1)\sigma(x_2)\frac{\delta^2}{\delta h(3)\delta h(4)}P_h(x_1x_2,12)\right]\Bigg|_{3=4=1,h=0}$$
$$=(\nabla_i^{(3)}\nabla_i^{(4)}\{G_2^{(0)}(13)_2^{(0)}(14)C_{40}(12)+G_2^{(0)}(23)G_2^{(0)}(24)C_{22}(12)$$
$$+[G_2^{(0)}(13)G_2^{(0)}(24)+G_2^{(0)}(23)G_2^{(0)}(14)]C_{32}(12)\})|_{3=4=1}.$$

Since

$$(\nabla_i^{(3)}G_2^{(0)}(13))|_{3=1}=(\nabla_i^{(4)}G_2^{(0)}(14))|_{4=1}=0,$$

we find

$$\overline{K}^{(0)}(12)=(\nabla_i^{(1)}G_2^{(0)}(12))^2C_{22}(12). \tag{226}$$

Putting these results together, using Eq. (209),

$$f(x)\gamma(x)=\tan\left[\frac{\pi}{2}F(x)\right] \tag{227}$$

and

$$\nabla_xF(x)=\frac{2}{\pi}\gamma(x)\nabla_xf(x), \tag{228}$$

we obtain the scaling equation

$$\vec{x}\cdot\nabla F(x)+\nabla^2F(x)+\tan\left[\frac{\pi}{2}F(x)\right]\left[\frac{1}{\mu}-\frac{\pi}{2}(\nabla_xF(x))^2\right]=0, \tag{229}$$

where

$$\frac{\pi}{2\mu}=\frac{L^2\langle(\nabla m)^2\rangle_0}{S_2^{(0)}}. \tag{230}$$

Analyzing Eq. (229), however, one finds the remarkable result that there is an analytical solution given by

$$F=\frac{2}{\pi}\sin^{-1}[e^{-x^2/2}] \tag{231}$$

with μ given by the OJK result $\mu=\pi/2d$. Thus the equation of motion method gives the same result as the method of evaluating $C_\psi(12)$ directly. This lends strong support to the structure of the approach.

One interesting question concerns the work here compared to that in the TUG. If one makes the replacement

$$\frac{1}{\mu}-\frac{\pi}{2}(\nabla F)^2\rightarrow\frac{1}{\mu} \tag{232}$$

in Eq. (229), one obtains the basic equation in the TUG approach where μ must then be determined as the solution to a nonlinear eigenvalue problem. Given the selected μ^*, one then has the analytic results $\lambda=d-(\pi/4\mu^*)$ and $\nu=d-(\pi/2\mu^*)$. In the TUG approach λ and ν are not independent ($\nu=2\lambda-d$). The TUG results for λ and ν as functions of d are shown in Tables I and II. It seems clear that the TUG values for λ and ν are superior to those of the present second-order theory. In particular, the TUG approach gives the known exact results in one dimension. Clearly the behavior of ν with d is different in the two approximations with the TUG giving $\nu\rightarrow 0$ and the current theory giving $\nu\rightarrow\infty$ as d increases. Thus the TUG results are in agreement with the speculation of Bray and Humayun that large d corresponds to the OJK limit.

It should be kept in mind that while the TUG approach does well in giving these exponents, it is deficient in treating the smoothness of the auxiliary-field correlation function which enters into the determination of defect dynamics. The present theory gives good results for the exponents-the corrections to OJK all appear to be in the right direction, and give the auxiliary-field correlation functions which are smooth.

XII. CONCLUSIONS

The key accomplishment in this paper is to show how one can set up a systematic calculation which allows for both nontrivial nonequilibrium exponents λ and ν, and results for the auxiliary-field correlation function which are smooth enough to offer physical results for defect structures. The structure of the theory presented here is appealing since the deviation from the OJK results can be handled perturbatively, and one can see and control the post-Gaussian corrections.

The key question remaining is the uniqueness of the theory developed here. While it seems that there is a degree of universality in this problem, the answer to the question posed involves the robust nature of this universality. This can be investigated by looking at those changes in the equation of motion for the auxiliary field which may change scaling results. Thus one can try to find marginal variables which could lead to exponents which depend on a parameter. It seems reasonable that the uniqueness of the particular realization of the theory presented here can be investigated within its own structure.

It is clear that one can introduce more sophisticated resummation methods than the direct method used here. These methods could be useful, for example, in establishing the connection between the exponents governing the statistics of the auxiliary field and those governing the statistics of the order parameter field. While these are equal at $O(2)$, it is not at all obvious that this holds at higher order. It will be interesting to see if one can formulate alternatives to and resummations of the ϕ_p expansion developed here. For example it seems to desirable to find an expansion, as in the TUG, which matches the exact solution for $d=1$.

In the second paper in this series the theory is extended to the n-vector model. It will turn out, at least at $O(2)$, that the ϕ expansion is related to a large n expansion. The main focus of this second paper will be to look at the determination of defect spatial and velocity correlations.

ACKNOWLEDGMENTS

I thank Dr. R. Wickham for useful comments. This work was supported in part by the MRSEC Program of the National Science Foundation under Contract No. DMR-9400379.

APPENDIX: SOLUTION FOR $S_2^{(0)}(T)$

In this appendix we present the solution to the equation

$$S_2^{(0)}(t)=\exp\left(2\bar{\omega}\int_{t_0}^{t}\frac{d\tau}{S_2^{(0)}(\tau)}\right)\frac{S_2^{(0)}(t_0)}{Q^{d/2}(t)},$$

where

$$Q(t)=1+\alpha(t-t_0).$$

Cross multiply by $Q^{d/2}(t)$, and take the time derivative, to obtain the rather simple equation

$$\dot{S}_2^{(0)}(t)+\frac{d\alpha}{2}\frac{S_2^{(0)}(t)}{Q(t)}=2\bar{\omega}.$$

Introducing the integrating factor

$$S_2^{(0)}(t)=\exp\left(-\frac{d}{2}\int_{t_0}^{t}d\tau\frac{\alpha}{Q(\tau)}\right)X(t),$$

where

$$\dot{X}(t)=\exp\left(\frac{d}{2}\int_{t_0}^{t}d\tau\frac{\alpha}{Q(\tau)}\right)2\bar{\omega}.$$

This is easily integrated to obtain

$$X(t)=S_2^{(0)}(t_0)+2\bar{\omega}\int_{t_0}^{t}d\bar{t}\,\exp\left(\frac{d}{2}\int_{t_0}^{\bar{t}}d\tau\frac{\alpha}{Q(\tau)}\right).$$

We can then do the integral

$$\int_{t_0}^{t}d\tau\frac{\alpha}{Q(\tau)}=\alpha\int_{t_0}^{t}d\tau\frac{d}{d\tau}\ln[1+\alpha(\tau-t_0)]$$
$$=\ln[1+\alpha(t-t_0)].$$

This result is then inserted in the integrating factor and the remaining integral over $\bar{t}$ can easily be carried out to obtain the result given by Eq. (134).

[1] See A. J. Bray, Adv. Phys. **43**, 357 (1994) for a recent review.
[2] T. Ohta, D. Jasnow, and K. Kawasaki, Phys. Rev. Lett. **49**, 1223 (1982).
[3] G. F. Mazenko, Phys. Rev. B **42**, 4487 (1990). TUG stands for ''theory of unstable growth.''
[4] A. J. Bray and K. Humayun, Phys. Rev. E **48**, R1609 (1993).
[5] G. F. Mazenko, Phys. Rev. E **49**, 3717 (1994).
[6] R. A. Wickham and G. F. Mazenko, Phys. Rev. E **55**, 2300 (1997).
[7] A preliminary version of this work is available at cond-mat/9711029.
[8] G. F. Mazenko and R. A. Wickham, Phys. Rev. E **55**, 1321 (1997).
[9] F. Liu and G. F. Mazenko, Phys. Rev. B **46**, 5963 (1992).
[10] G. F. Mazenko and R. A. Wickham, Phys. Rev. E **55**, 5113 (1997).
[11] G. F. Mazenko, Phys. Rev. Lett. **78**, 401 (1997).
[12] G. F. Mazenko, Phys. Rev. E **56**, 2757 (1997).
[13] G. F. Mazenko and R. A. Wickham, Phys. Rev. E **57**, 2539 (1998).
[14] J. W. Cahn and J. E. Hilliard, J. Chem. Phys. **28**, 258 (1958).
[15] J. S. Langer, M. Bar-on, and H. D. Miller, Phys. Rev. A **11**, 1417 (1975).
[16] G. F. Mazenko, O. T. Valls, and M. Zannetti, Phys. Rev. B **38**, 520 (1988).
[17] Y. Oono and S. Puri, Phys. Lett. B **2**, 861 (1988).
[18] C. Yeung and D. Jasnow, Phys. Rev. B **42**, 10 523 (1990).
[19] S. Puri and C. Roland, Phys. Rev. A **151**, 500 (1990).
[20] C. Yeung, Y. Oono, and A. Shinozaki, Phys. Rev. E **49**, 2096 (1994).
[21] R. E. Blundell, A. J. Bray, and S. Sattler, Phys. Rev. E **48**, 2476 (1993).
[22] R. E. Blundell and A. J. Bray, Phys. Rev. E **49**, 4925 (1994).
[23] There has been speculation (see Ref. [1]), that the OJK result is obtained in a systematic expansion in powers of $1/d$. We do not find support for this conjecture here.
[24] The nature of the expansion is discussed below. Convergence properties, if any, remain to be investigated.
[25] D. S. Fisher and D. A. Huse, Phys. Rev. B **38**, 373 (1988).
[26] F. Liu and G. F. Mazenko, Phys. Rev. B **44**, 9185 (1991).
[27] G. Porod, in *Small Angle X-Ray Scattering*, edited by O. Glatter and L. Kratky (Academic , New York, 1983).
[28] H. Tomita, Prog. Theor. Phys. **72**, 656 (1984); **75**, 482 (1986).
[29] The exponent ν was introduced in the TUG. However, within the approximation used there, $\nu=d-2\lambda$ and ν did not appear to be independent. See R. A. Wickham and G. F. Mazenko, Phys. Rev. E **55**, 2300 (1997). The work here indicates that ν is independent of λ.
[30] The perturbation expansion developed here does not appear to be uniformly useful as a function of dimensionality. As discussed in Sec. XI, the TUG approximation does much better in giving the exact results for λ and ν in one dimension compared with the perturbation theory results presented here. For large d, where λ approaches the OJK result, the exponent ν increases and the algebraic component of the scaling function differs qualitatively from the OJK result. Since we do not know the large d solution to our problem for ν, we do not know whether this clearly nonperturbative result is valid or not. Our prejudice is that it is an artifact of the approximation and ν should go to zero for large d as in the TUG.
[31] In the TUG it was assumed that m is governed by a Gaussian probability, distribution and the variance is determined by demanding that the equation of motion satisfied by the order parameter by, on average,
$$\langle\Lambda(1)\psi(1)\psi(2)\rangle=-\langle V'[\psi(1)]\psi(2)\rangle.$$
[32] Bray and Humayun argue that this procedure can be made systematic by adding a *color* index $\alpha(1,2,\ldots,N)$ to the field $m\rightarrow m_\alpha$ and generalize the nonlinearity in their theory to
$$(\nabla m)^2\rightarrow\frac{1}{N}\sum_{\alpha=1}^{N}(\nabla m_\alpha)^2,$$
and expanding in powers of $1/N$. Similarly, they suggest that one should be able to reformulate this process as an expansion in powers of $1/d$.
[33] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. **A8**, 423 (1973).
[34] C. de Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978). See, relating to the current application, G. F. Mazenko, O. T. Valls, and M. Zannetti, Phys. Rev. B **38**, 520 (1988).
[35] This is one of the arguments in favor of the universality of the results obtained.
[36] One point to keep in mind is the well known property of the MSR method that the equal-time correlation functions with auxiliary fields must vanish if one is to satisfy causality: $G_{Mm}(11)=0$.